PartIA Vector and Matrices

zc231

Each question will be labeled in the form $\alpha, \beta\gamma$ where $\alpha \in \{1, 2, 3, 4\}$ represents the paper number, $\beta\gamma$ represents the question number in that paper. For example, 1,11G means question 11G in paper 1. I will omit the proofs in the notes or book work. The solutions provided might not be the best ways to solve the problems and if you find any mistakes or if you have any elegant ways of solving some of the problems please email me at zc231@cam.ac.uk.

2009

1,1C (a): $\Re(z\bar{\alpha}) = 0$ and so $arg(z) = arg(\alpha)$ which is a line. (b): Square both sides and we have $4|z|^2 = (z + \bar{z})^2$ which then gives $(z - \bar{z})^2 = 0$ so z is real. (c): Let $z = re^{i\theta}$ one gets $\log r = \theta$ which defines a logarithmic spiral.

1,2B $B = A^{-1}A^+$ is unitary if and only if $BB^+ = A^{-1}A^+A(A^{-1})^+ = I$ if and only if $A^+A = AA^+$. For (ii), $|Cx| = 0$ if and only if $x^+C^+Cx = 0$ and as C is normal so this holds if and only if $x^+CC^+x = 0$ which is $|C^+x|^2 = 0$. Let $De = \lambda e$ so $(D - \lambda)e = 0$ and by (ii) we know $(D^+ - \lambda^*)e = 0$ so e is eigenvector of D^+ with eigenvalue λ^*.

1,5C $(a\times b)\cdot(a\times c) = \epsilon_{ijk}a_jb_k\epsilon_{ilm}a_lc_m$ and as $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ and so we have $|a|^2 b\cdot c - (a\cdot c)(a\cdot b)$ and use $|a|^2 = 1$.

The ith component of $(a \times b) \times (a \times c)$ is

$$\epsilon_{ijk}(a\times b)_j(a\times c)_k = \epsilon_{ijk}\epsilon_{jlm}a_lb_m\epsilon_{kpq}a_pc_q = (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})\epsilon_{kpq}a_lb_ma_pc_q = \epsilon_{lpq}a_lb_ia_pc_q - \epsilon_{mpq}a_ib_ma_pc_q$$

and the first term above is the ith term of $[a \cdot (a \times c)] \cdot b$ which is 0, and the second one is

$$-\epsilon_{lpq}a_ib_la_pc_q = \epsilon_{plq}a_pb_lc_qa_i = [a \cdot (b \times c)] \cdot a_i.$$

Then as $a \cdot b = \cos\theta$ where θ is the angle between a, b, and $\delta(A, B) = \cos\theta$ we have $a \cdot b = \cos\delta(A, B)$.

The next part is clear if you follow the hint (you probably need to draw a picture to illustrate this, so you can, without loss of generality by rotation put B, C on the horizontal plane of the sphere) and the angle α is the same as the angle between the normals.

We have by definition and the fact $|a| = |b| = 1$, that $\sin\delta(A, B) = |a \times b|$. Therefore,

$$\cos\delta(A, B)\cos\delta(A, C) + \sin\delta(A, B)\sin\delta(A, C)\cos\alpha = (a \cdot b)(a \cdot c) + |a \times b||a \times c|\cos\alpha$$

and use the expression for $\cos\alpha$ we conclude this gives

$$(a \cdot b)(a \cdot c) + (a \times b) \cdot (a \times c) = b \cdot c = \cos(B, C)$$

by (i). Similarly,

$$\sin\alpha = \frac{|(a \times b) \cdot (a \times c)|}{|a \times b||a \times c|}$$

and so

$$\sin\alpha\sin\delta(A,C)-\sin\beta\sin\delta(B,C)=\frac{|(a\times b)\times(a\times c)|}{|a\times b|}-\frac{|(b\times c)\times(b\times a)|}{|b\times a|}$$

and use (ii) we have

$$\frac{|a\cdot(b\times c)||a|}{|a\times b|}-\frac{|b\cdot(a\times c)||b|}{|a\times b|}=0$$

because $|a|=|b|=1$ and $|a\cdot(b\times c)|=|b\cdot(a\times c)|$. By symmetry we conclude the three equalities hold.

It is clear that $\alpha=\beta=\gamma$ because we have $\sin\alpha=\sin\beta=\sin\gamma$ from the second set of equalities and we also have three equalities by permuting A,B,C and use the first equality involving $\cos\alpha$ so we have $\cos\alpha=\cos\beta=\cos\gamma$ and so $\alpha=\beta=\gamma$. Then if we set $\delta(A,B)=\delta(B,C)=\delta(A,C)=t$ we have

$$\cos\alpha\sin^2 t+\cos^2 t=\cos t$$

and writing $\sin^2 t=1-\cos^2 t$ we have

$$\cos\alpha=\frac{\cos t(1-\cos t)}{1-\cos^2 t}=\frac{\cos t}{1+\cos t}$$

because $\cos t\neq 1$ (otherwise we have $\cos\alpha=0$ which then implies all three angles are $\pi/2$ which is impossible from basic geometry) and as $1>\cos t$ so $\cos\alpha<\frac{1}{2}$ so $\alpha>\pi/3$.

1,6B $Ax=c$ basically solves the intersection of three hyperplanes in $\mathbb{R}^3$ so we have either a point (one solution), a line or a plane (infinitely many solutions) or empty (no solution). If $\det A\neq 0$ we have a unique solution. If $\det A=0$ and $c\notin$ the image of A then we have no solution and if $\det A=0$ and c is in the image we have infinitely many solutions. If $\det A=0$ then we have y (or y_1,y_2 in the case when A has two eigenvectors of eigenvalue 0) then if x_1,x_2 are solutions we have some λ_1,λ_2 such that $x_1+\lambda_1 y_1+\lambda_2 y_2=x_2$.

Determinant is $(a-b)^2(2a+b)$. If $a\neq b, 2a+b\neq 0$ then the kernel is trivial and the image is $\mathbb{R}^3$. If $a=b$ not zero, then the kernel is the plane $x_1+x_2+x_3=0$ and the image is the line $x_1=x_2=x_3$. If $a=b=0$ then the kernel is $\mathbb{R}^3$ and the image is $\{0\}$. If $2a+b=0$ and $a,b\neq 0$ then the kernel is the line $x_1=x_2=x_3$ and the image is the plane $x_1+x_2+x_3=0$. If $a\neq b, 2a+b\neq 0$ then we have a unique solution. If $a=b$ not zero, then we have a solution if only if $c=1$ and the solution is the plane $a(x_1+x_2+x_3)=1$. If $a=b=0$ then there is no solution. If $b=-2a$ not zero we have a solution if and only if $c=-2$ and the solution is the line $x_2=x_3=x_1-\frac{1}{a}$.

1,7A Let $Av=\lambda v$ then $v^*Av=\lambda|v|^2=v^*A^+v=(v^*Av)^+=\lambda^*|v|^2$ so $\lambda=\lambda^*$.

For (a): Take $Av_1=\lambda_1 v_1, Av_2=\lambda_2 v_2$ so

$$v_2^+Av_1=\lambda_1 v_2^+v_1, v_2^+Av_1=(Av_2)^+v_1=\lambda_2^*v_2^+v_1=\lambda_2 v_2^+v_1$$

and as $\lambda_1\neq\lambda_2$ so we have $v_2^+v_1=0$. For (b) as A is Hermitian with distinct eigenvalues you can diagonalise it and so the polynomial equation in matrix can be considered as individual equations on the diagonalised matrix so by definition we have 0 as each eigenvalue is a root of the characteristic polynomial.

For (c) diagonalise A by unitary matrix U so we have

$$\frac{x^+Ax}{x^+x}=\frac{x^+U^+DUx}{(Ux)^+Ux}=\frac{y^+Dy}{y^+y}$$

because $|Ux| = |x|$ for any unitary matrix and write $y = Ux$. Therefore, we see by direct computation this is bounded above by the maximum of eigenvalue and bounded below by the minimum of eigenvalue.

For (d) for example consider the matrix

$$B = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

so both eigenvalues are zero but pick $x = (2, 1)$ then if such C exists we have $\frac{3}{5} \leq 0$.

1,8A (a) is book work (you should know what isometry looks like, by using rotation and reflection).

For (b), we need to classify matrices A satisfying the condition that $\eta = A^t \eta A$ where

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Taking entries a, b, c, d for A we conclude that

$$a^2 - c^2 = 1, ab = cd, b^2 - d^2 = -1.$$

Then this is very similar as in (a), except we replace $\cos\theta, \sin\theta$ etc. by $\cosh\theta, \sinh\theta$.

For (c), we need our matrices to satisfy $a^2 - c^2 = 1$ and $a^2 + c^2 = 1$ so we have $c = 0$ and similarly we have $d^2 - b^2 = 1$ and $d^2 + b^2 = 1$ so $b = 0$. Therefore we have $a^2 = d^2 = 1$.

2010

1,1A Book work (change of basis matrix).

1,2C $z_1 = \zeta_3^i$ and $z_2 = \zeta_9^j 2$ for some i, j. In fact you can fix i and vary j, because if you replace z_1 by $\zeta_3 z_1$ and z_2 by $\zeta_3 z_2$ the modulus is unchanged. (b) square both sides so we compare $z_1\bar{z}_2 + \bar{z}_1 z_2$ and $2|z_1 z_2|$ and the previous one is twice the real part of $z_1\bar{z}_2$ and the second one is twice the modulus of $z_1\bar{z}_2$. No, for example $z_1 = 1, z_2 = i$.

1,5A By suffix notation we have

$$tr(B^T A) = (B^T)_{ij} A_{ji} = B_{ji} A_{ji} = A_{ji} B_{ji} = (A^T B)_{ii} = tr(A^T B).$$

We have $tr(A^T A) = (A^T)_{ij} A_{ji} = A_{ji} A_{ji}$ which is $\sum_{i=1,j=1} A_{ij}^2 \geq 0$ and equality holds if and only if $A_{ij} = 0$ for all i, j. Then one defines a norm by

$$\langle A, B \rangle = tr(B^T A)$$

which is an inner product space and the result follows by Cauchy-Schwarz (this can be proved directly by suffix notation and Cauchy-Schwarz) and equality holds if and only if A or B is zero matrix or B is a scaler multiple of A (linearly dependent). For the last part we need $a+2c+b = 0, a+b-2d = 0, c+d = 0$ by direct computation and so we have $a+b = 2d, c = -d$ which is 2-dimensional (as $2C = A - B$) and we can pick basis $(2, 0, -1, 1)$ and $(0, 2, -1, 1)$.

1,6C Recall the determinant of a matrix with rows (or columns) a, b, c is given by $a \cdot (b \times c)$ and note that $S = AB$ where

$$A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}$$

where a_1, a_2, a_3 are row vectors and b_1, b_2, b_3 are column vectors. Therefore,

$$\det(S) = \det(A)\det(B) = (a_1 \cdot a_2 \times a_3)(b_1 \cdot b_2 \times b_3).$$

S has maximal rank if and only if determinant of S is non-zero and by above observation this holds if and only if both A, B have non-zero determinant, i.e. $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are both linearly independent sets.

So in general if $T_{ij} = c_i \cdot d_j$ then the matrix $T = CD$ where C has rows $c_1, \ldots, c_n$ and D has columns $d_1, \ldots, d_n$ and so T has maximal rank if and only if C, D have non-zero determinants which is the same as saying $\{c_1, \ldots, c_n\}, \{d_1, \ldots, d_n\}$ are both linearly independent. For the last part a direct argument would be pick a, b linearly independent and take $a+b, a+2b, a+3b$ etc. and show that any three are linearly dependent. Alternatively, start with two linearly independent vectors a_1, a_2 so they span a plane S. Given $a_1, \ldots, a_i$, $i \geq 2$ pick a_{i+1} which are not parallel to $a_1, \ldots, a_i$ and a_{i+1} lies in S then clearly by construction any two vectors are linearly independent but any three will be dependent because they span a plane.

1,7B (i), (ii) are straightforward (by writing down the definition). (iii) follows from

$$\chi_{A^2}(t^2) = \det(A^2 - t^2 I) = \det((A - tI)(A + tI)) = \det(A - tI)\det(A + tI) = \chi_A(t)\chi_A(-t).$$

For (iv) we have

$$\chi_{A^{-1}}(t) = \det(A^{-1} - tI) = \det(A^{-1})\det(I - tA)$$

and for $\det(I - tA)$ we take out the factor $-t$ so $\det(I - tA) = (-t)^n \det(A - t^{-1}I)$ and so we conclude

$$\chi_{A^{-1}}(t) = \det(A^{-1})(-t)^n \chi_A(t^{-1}).$$

λ is a root of $\chi_A(t)$ if and only if λ is an eigenvalue. It is clear that if λ is an eigenvalue then we have non-zero v with $(A - \lambda I)v = 0$ so $\det(A - \lambda I) = 0$. Conversely, if $\det(A - \lambda I) = 0$ then the kernel is non-trivial so there exists v with $(A - \lambda I)v = 0$ and so $Av = \lambda v$.

By (iii) we compute

$$\chi_{A^4}(t^4) = \chi_{A^2}(t^2)\chi_{A^2}(-t^2) = \chi_A(t)\chi_A(-t)\chi_A(it)\chi_A(-it)$$

and if we set $t^4 = z$ then if $z = \mu$ is a root for $\chi_{A^4}(z)$ (which means z is an eigenvalue) then we see at least one of

$$\chi_A(\lambda), \chi_A(-\lambda), \chi_A(i\lambda), \chi_A(-i\lambda)$$

vanishes where λ is any choice of fourth root of μ and so we conclude A has some eigenvalue λ with $\lambda^4 = \mu$.

1,8B Let $Rv = \lambda v$ and so $v^+R^+Rv = |\lambda|^2 v^+v$ and as λ, v are both real and $R^+R = I$ so we have $\lambda = \pm 1$. It represents an isometry (as $|\lambda| = 1$).

For N the only v with $\lambda = 1$ is a multiple of $(1, 1, 1)$. For M we have v being a multiple of $(1, 0, 0)$. For P the eigenspace is spanned by $v_1 = (1, 0, -1), v_2 = (0, 1, -1)$. If a matrix represents a rotation then the determinant must be 1 as it is orientation preserving and so it is either M or N and the axis of rotation is the eigenvector with eigenvalue 1. Here is a method one might be willing to use to check whether a given matrix is a rotation. Take the vector fixed by the matrix and take a basis for $\mathbb{R}^3$ containing the fixed vector and two other vectors such that the three vectors are mutually perpendicular. Then find the images of the other two vectors and compute the angles between each of them and the image (by using dot product) and if we get the same angle then it must be a rotation and if not then it is not a rotation (consider the action on the basis). So in this case we conclude M is a rotation about $(1, 1, 1)$ by $2\pi/3$. The last matrix is a reflection because any reflection must have determinant -1. To check it is a reflection about the plane spanned by $(1, 0, -1)$ and $(0, 1, -1)$ we take a vector which is perpendicular to this plane, say $(1, 1, 1)$ and the image of $(1, 1, 1)$ under P is $(-1, -1, -1) = -(1, 1, 1)$ and so it is a reflection.

Clearly P is real symmetric so it is diagonalisable over $\mathbb{R}$. M has distinct eigenvalues and it has non-real eigenvalues so it is only diagonalisable over $\mathbb{C}$. For N the only eigenvalue is 1 and the dimension of eigenspace is 1 so it is not diagonalisable.

2011

1,1C (i) $\exp(i\log(z)) = 1$ so $\log(z)$ is a multiple of 2π which means z is real and $z = \exp(2n\pi)$. (ii) As $\log\bar{z} = \overline{\log z}$ so we have

$$z^i + \bar{z}^i = \exp(i\log z) + \exp(i\overline{\log z})$$

and expand $\log z = \log|z| + \theta$ we conclude that

$$\cosh\theta = \exp(\frac{\pi i}{2} - i\log|z|)$$

which means $\cosh\theta = 1$ (as $\cosh\theta \geq 1$ but RHS has modulus 1) and so $\theta = 0$ and we also have $\log|z| = \frac{\pi}{2} + 2n\pi$ so z is real and $z = \frac{\pi}{2} + 2n\pi$. For the last one we need $\log|z| = \theta$ which is a spiral.

1,2A Standard change of basis.

1,5C (1) is a straight line through a with direction vector b. For (2) note if x_1, x_2 are both solutions then $(x_1 - x_2) \times c = 0$ so $x_1 - x_2$ is parallel to c so take any fixed point x_0 which satisfy the equation then we have $x = x_0 + \lambda c$ ($c \cdot d = 0$ is a necessary condition for solution to exist as if you dot both sides by c you will get $c \cdot d = 0$) and to see at least one solution exists let x_0 be a vector such that c, d, x_0 are mutually perpendicular and scale x_0 so that $|x_0||c| = |d|$ and so in (2) c is the direction vector and the line must be perpendicular to d.

By the above observation we conclude (2) satisfies an equation of the form (1) with $a = \frac{c\times d}{|c|^2}$ and $b = c$ so that $a \cdot b = 0$ and $|b| = |c|$. Clearly by defining a in this way a is a solution. For (1) find λ with $(y - (a + \lambda b)) \cdot b = 0$ so they are perpendicular, and we have $\lambda = \frac{y\cdot b - a\cdot b}{|b|^2}$ and then the point we want is just $a + \lambda b$ for this λ. The method is similar for (2) as we can write (2) in terms of $x = \frac{c\times d}{|c|^2} + \lambda c$.

For the last part, one way to do this is to take a basis $\{m, n, m \times n\}$ for $\mathbb{R}^3$ (the condition $m \times n \neq 0$ implies m, n are linearly independent). Then let $x = x_1 m + x_2 n + x_3(m \times n)$ be the most general form and let $x \cdot m = \mu, x \cdot n = \nu$ we then have

$$x_1 = \frac{\nu r - \mu}{r^2 - 1}, x_2 = \frac{\mu r - \nu}{r^2 - 1}$$

where $r = m \cdot n$ is a constant and $r^2 \neq 1$ because m is not parallel to n. Therefore we can put this into the form of (1) by setting $a = x_1 m + x_2 n$ and $b = m \times n$ and $\lambda = x_3$. To put it into (2) we simply note

$$x \times (m \times n) = a \times (m \times n)$$

so pick $c = (m \times n)$ and $d = a \times (m \times n)$ for the above a.

1,6A Let $y = \Phi(x) = x + (\alpha - 1)(n \cdot x)n$, then $y \cdot n = \alpha(n \cdot x)$ and also we have

$$n \times y = n \times x, n \times (n \times y) = n \times (n \times x)$$

and so we have $(n \cdot y)n - y = (n \cdot x)n - x$ and if $\alpha \neq 0$ we have

$$x = (n \cdot y)n(\frac{1}{\alpha} - 1) + y$$

and so the inverse exists for $\alpha \neq 0$. When $\alpha = 0$ the inverse does not exist because $\Phi(x) = \Phi(x + \lambda n)$ for all $\lambda \in \mathbb{R}$ so the map is not injective and so not invertible.

When Φ is not invertible we have $\alpha = 0$ and $\Phi(x) = x - (n \cdot x)n$. For any x we can write $x = \lambda n + \mu v$ where v is perpendicular to n and thus

$$\Phi(x) = \lambda n + \mu v - \lambda n = \mu v$$

and therefore we conclude that the image is the plane spanned by vectors perpendicular to n. The kernel is then clearly the line spanned by n because we need $\mu v = 0$ and hence $\mu = 0$.

Let $y = \Phi(x) = x + (\alpha - 1)(n \cdot x)n$ and so by suffix notation we write

$$y_i = x_i + (\alpha - 1)n_j x_j n_i = (\delta_{ij} + (\alpha - 1)n_i n_j)x_j$$

and so $A_{ij} = \delta_{ij} + (\alpha - 1)n_i n_j$. When Φ is invertible the inverse is

$$x = (n \cdot y)n(\frac{1}{\alpha} - 1) + y$$

which gives

$$x_i = n_j y_j n_i(\frac{1}{\alpha} - 1) + y_i = (n_i n_j(\frac{1}{\alpha} - 1) + \delta_{ij})y_j$$

and so $B_{ij} = n_i n_j(\frac{1}{\alpha} - 1) + \delta_{ij}$.
When $n = \frac{1}{\sqrt{3}}(1, 1, 1)$ we have $A_{ij} = \delta_{ij} + \frac{1}{3}(\alpha - 1)$. So we have

$$C^{-1}Ax = x, \text{ where} C^{-1}A = \frac{1}{3}\begin{pmatrix} a+1 & a-2 & a+1 \\ a+1 & a+1 & a-2 \\ a-2 & a+1 & a+1 \end{pmatrix}$$

and we compute the characteristic polynomial of $C^{-1}A$ which is

$$t^3 + (-\alpha - 1)t^2 + (\alpha + 1)t - \alpha = (t - \alpha)(t^2 - t + 1).$$

If $C^{-1}Ax = x$ then x is an eigenvector with eigenvalue 1 and hence we must have $\alpha = 1$ so for all $\alpha \neq 1$ such x does not exist. For $\alpha = 1$, we have A being the identity matrix and so $Cx = x$. Then x is a multiple of eigenvector of C with eigenvalue 1, which is generated by $(1, 1, 1)$. Finally by considering the images $(1, 0, 0)$ and $(0, 1, 0)$ under C and take the dot product between the image and the preimage we conclude that C is a rotation about x by θ where $\cos\theta = \frac{2}{3}$.

1,7B The characteristic polynomial factorises into $(t-1)(t-2)(t-3)$. The eigenvectors are $(1, 0, -1)$, $(1, 1, 0)$ and $(1, 1, 1)$ for $t = 1, 2, 3$ respectively. (b) is book work (argue by determinant and whether x lies in the image of A when $\det A = 0$).

For the last part, as 0 is not an eigenvalue so the solutions for $\lambda = 0$ is just $M^{-1}b$ which is unique. For $\lambda = 1$, the image of $M - I$ is the plane $x_1 = x_2$ and b is not in the image so we have no solution. For $M - 2I$ the image is the plane $x3 - x1 = 2(x2 - x1)$ and b is in the image so we find one solution $y = (0, 1, 4)$ and thus $x = y + \lambda(1, 1, 0)$ for all $\lambda \in \mathbb{R}$.

1,8B (a) (i) Take $Mv = \lambda v$ and so $|\lambda|^2|v|^2 = v^+M^+Mv = v^+MMv = \lambda^2|v|^2$ so λ is real. (ii) If $Mv = \lambda v$ then $Mv^* = \lambda v^*$ as M and λ are real, and so pick $v + v^*$ which is real. (iii) Let $Mv_1 = \lambda_1 v_1, Mv_2 = \lambda_2 v_2$ where $\lambda_1 \neq \lambda_2$. Then

$$\lambda_1 v_1^+ v_2 = v_1^+ M^+ v_2 = v_1^+ M v_2 = \lambda_2 v_1^+ v_2$$

so $v_1^+ v_2 = 0$.

Is A is antisymmetric then $A^t = -A$ and so $(A^2)^t = (A^t)^2 = (-A)^2 = A^2$. So A^2 is symmetric and hence each eigenvalue is real. By studying the characteristic polynomial of A^2, we see that each eigenvalue of A^2 must be a square of some eigenvalue of A and so it suffices to prove that each eigenvalue of A is pure imaginary. Let $Av = \lambda v$ so

$$-|\lambda|^2|v|^2 = -v^+A^+Av = v^+A^2v = \lambda^2|v|^2$$

and so $\lambda^2 = -|\lambda|^2 \leq 0$.

We know at least one of $i\lambda$ or $-i\lambda$ is eigenvalue of A but as they are roots of characteristic polynomial so the roots appear as complex conjugates and so both of them are eigenvalues of A. Thus, pick u_1, u_2, w_1, w_2 to be eigenvectors of $i\lambda, -i\lambda, i\mu, -i\mu$ respectively. As $Au = i\lambda u_1$ so if we take complex conjugates we have $Au_1^* = -i\lambda u_1^*$ and so we conclude that we can choose u_1, u_2 in the way that $u_1^* = u_2$ and similarly pick $w_1^* = w_2$. Now take $u = u_1 + u_2, u' = i(u_1 - u_2), w = w_1 + w_2, w' = i(w_1 - w_2)$ which are all real vectors. Then we have

$$Au = \lambda u', Au' = -\lambda u, Aw = \mu w', Aw' = -\mu w$$

and also

$$\lambda u'^+u = u^+Au = (u^+Au)^+ = u^+A^+u = -u^+Au = -\lambda u^+u$$

and as $\lambda \neq 0$ so $u'^+u = 0$. Similarly we have $w'^+w = 0$. Finally, by (b) u_1, u_2 are eigenvectors of eigenvalue $-\lambda^2$ and w_1, w_2 are eigenvectors of eigenvalue $-\mu^2$ and since they are distinct so by (a) we conclude that u_1, u_2 are orthogonal to w_1, w_2 and hence u, u' are orthogonal to w, w'.

2012

1,1C (a) It is a straight line. $\exp(z) = \exp(1+it)$ so the image has modulus $\exp(1)$. (b)(i) $z-1 = 2i\pi n$ (ii) We have $\log z = \pm\frac{i\pi}{2}$ and so $|z| = 1$ and $\theta = \pm\frac{\pi}{2}$. (c) multiply both sides by π and take exponential we have

$$\exp(|z|\pi) = -\exp(-z\pi)$$

and then consider the modulus of the above (and write $z = x + iy$) we need $z \in \mathbb{R}$ but then we have $|z| - z = 0$.

1,2A A is rotation. For B we find the eigenvectors $u = (1, 1 + \sqrt{2})$, $v = (1, 1 - \sqrt{2})$. and so B is dilation in the direction u and v. By direction computation

$$C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and so it is a shear.

1,5C Distance is d as n is unit vector. If $|p \cdot n - d| = r$ then $|p \cdot n - x \cdot n| = r$ for any x on the plane and so $|(x - p) \cdot n| = r$. If x also lies on the sphere then $|x - p| = r$. This shows that $x - p$ must be parallel to n then the sphere S is perpendicular to the plane and hence there is only one point of intersection. If you want an algebraic explanation, suppose $p \cdot n - d = r$ then we take $x = p - rn$ and so $x \cdot n = d$ and $|x - p| = r$, and if $p \cdot n - d = -r$ we take $x = p + rn$. If we have two points of intersections x, y then $x \cdot n = y \cdot n$ and also $x - p, y - p$ must be parallel to n so $(x - p) \times n = (y - p) \times n$ so

$$(x - y) \cdot n = 0, (x - y) \times n = 0$$

and so $|x - y| = 0$.

a, b, c are linearly independent and positively oriented. For the second part the condition is actually necessary and sufficient so suppose we only have one point of intersection, then we must have $p - x$ parallel to n and so $|(p - x) \cdot n| = |p - x| = r$. Thus from this observation, if we write $p = \lambda a + \mu b + \nu c$ then the equation for the plane OAB is $x \cdot n = 0$ where $n = \frac{a \times b}{|a \times b|}$ and so we have $|p \cdot n| = r$. Since a, b, c are positive oriented so we have $p \cdot n = r$ and this gives $\nu c \cdot (a \times b) = |a \times b| r$ and so

$$\nu = r\frac{|a \times b|}{c \cdot (a \times b)} = r\frac{|a \times b|}{a \cdot (b \times c)}.$$

By a similar argument we can compute λ and μ and so the result follows.

For the last part the equation has the form $x \cdot n = d$ where n is a unit vector which is parallel to $a \times b + b \times c + c \times a$ and $|d|$ is the distance. Therefore, we can find d by evaluating the equation at any x on the plane. So for example we can take a, b or c and so we find $d = a \cdot (b \times c) > 0$ so the distance is $a \cdot (b \times c)$.

1,6A The first part is book work. For the second part the condition we need is A^{-1} exists. If A^{-1} exists then $X = A^{-1}B$ which exists and is unique. Suppose A^{-1} does not exist, then A has non-trivial kernel say $v \neq 0$ and $Av = 0$. Then if there exists X with $AX = B$, say $X = (x_1, x_2, x_3)$ where x_1, x_2, x_3 are column vectors, then $X' = (x_1 + v, x_2, x_3)$ also satisfies $AX' = B$ and $X' \neq X$. For the last part A is invertible so $X = A^{-1}B$.

1,7B M has eigenvalues 2 with multiplicity 2 and 3 with multiplicity 1 and when $\lambda = 2$ the dimension of eigenspace is 1, spanned by $(1, 0, 0)$ and so M is not diagonalisable. (b) follows from the

$$\det(P^{-1}AP - tI) = \det(P^{-1})\det(A - tI)\det(P) = \det(A - tI)$$

where $B = P^{-1}AP$ and so they have the same characteristic polynomial hence same eigenvalues with the same multiplicity. The converse is not true for example take

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and they are not similar because A is only similar to itself.

Every matrix is similar to an upper triangular matrix. If you don't know about this fact then any complex 2×2 matrix has at least one eigenvector v so we pick new basis to be $\{v, u\}$ where u is any vector linearly independent to v so the entry $c = 0$. Now the characteristic polynomial becomes $(t - a)(t - d)$. Then $(A - aI)(A - dI) = 0$ which is direct computation. For the last part, consider $Bv = \lambda v$ for some eigenvalue λ, then $B^k v = \lambda^k v$ and so $\lambda^k = 0$. This shows that every eigenvalue of B is 0, and so the characteristic polynomial is t^n as it has degree n. Therefore by Cayley-Hamilton theorem $B^n = 0$.

1,8B The eigenvalues are $1, 2, 4$ and the corresponding eigenvectors are $(-1, 1, 1)$, $(0, 1, -1)$, $(2, 1, 1)$. Multiply the equation by 2 we have

$$4x^2 + (x + 2y)^2 + (x + 2z)^2 = 2$$

which is an equation for ellipsoid. So the linear transformation should be

$$x' = \sqrt{2}x, y' = \frac{1}{\sqrt{2}}(x + 2y), z' = \frac{1}{\sqrt{2}}(x + 2z).$$

For (b)(i), let P be real orthogonal we have

$$\langle u, v \rangle = u^+ v = u^+ P^+ P v = (Pu)^+ P v = \langle Pu, Pv \rangle.$$

For (ii) let $Pv = \lambda v$, and so by (i) we have $|v| = |Pv| = |\lambda||v|$ so $|\lambda| = 1$. Also roots of characteristic polynomial appear as complex conjugates so λ^* is also an eigenvalue (as P is real).

As $\lambda = 1$ has multiplicity 2 so the other eigenvalue is either 1 or -1. If it is 1 then Q is similar to the identity which fixes everything. If the last eigenvector is -1, then it is similar to the diagonal matrix with entries $1, 1, -1$ and since similar matrices represent the same linear map (as they only differ by change of basis) so we conclude this is a reflection about a plane.

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1,1C $\exp(i\theta) = \cos\theta + i\sin\theta$ so we need $\sin n\theta = 0, \cos n\theta = 1$ and so $\theta = \frac{2j\pi}{n}, j = 1, 2, \ldots, n$. $z^3 = -8$ implies $z = 1 - \sqrt{-3}, 1 + \sqrt{-3}, -2$. (b) is a circle we expand both sides we have

$$|z - \frac{3}{8}i| = \frac{9}{8}.$$

1,2A (i) $b_{31} = R_{131}A_{11}R_{132}A_{21}R_{133}A_{31} = 0$ and so we need $\tau\theta_1 = -\frac{A_{31}}{A_{21}}$. For the next part $c_{31} = 0$ from the definition of matrix multiplication. Then we need $\tan\theta_2 = -\frac{b_{21}}{b_{11}}$. For θ_3, we need $\tan\theta_3 = -\frac{c_{32}}{c_{22}}$. (iv) follows from (i),(ii),(iii) and take appropriate θ_i as D is upper triangular.

1,5C If x, y are independent then the dimension is 2 and otherwise the dimension is 1. Then the proof is book work (triangular inequality follows from Cauchy-Schwarz if you expand the square of modulus in terms of sum of squares). Equality holds if and only if x, y are linearly dependent. As x, y are fixed then it suffices to consider the minimum of $z \cdot x + z \cdot y = z \cdot (x + y)$. Certainly we can change the direction of z so the minimum has the same modulus as the maximum so we consider $|z \cdot (x + y)|$ and so by Cauchy-Schwarz we conclude $z = \lambda(x + y)$ for some $\lambda \in \mathbb{R}$.

(i) Clearly $\lambda < 0$ but we need $|z| = 1$. As $|x + y| \leq |x| + |y| = 2$ so we have $\lambda \leq -\frac{1}{2}$. (ii) Finally suppose $x \cdot y = \cos\frac{2\pi}{3}$ then

$$|x + y|^2 = (x + y) \cdot (x + y) = |x|^2 + 2x \cdot y + |y|^2 = 1$$

and so $|x + y| = 1$. Thus we have $\lambda = -1$ and hence $S = -\frac{3}{2}$.

1,6A The first several parts are book work. The image of β is the plane spanned by $(1, 2, 3)$ and $(6, 4, 2)$ and $(1, -2, 1)$ is not in the image (by setting $\lambda(1, 23) + \mu(6, 4, 2)$ and check no λ, μ give $(1, -2, 1)$). For the last part it is clear that γ is surjective so the kernel is a dimensional one subspace. Any three vectors in $\mathbb{R}^2$ must satisfy a linear relation and in this case we have

$$2(1, 3) + (-2, 1) - 7(0, 1) = (0, 0)$$

and hence

$$2(1, 2, 0) + (0, 0, 1) - 7(0, 1, 0) = (2, -3, 1)$$

is in the kernel and the kernel is spanned by this vector (It is not necessary to put γ into a matrix every time to find the kernel).

1,7B (a) is book work (consider if they satisfy some linear relation pick the one with the least length). For (b) we complete the squares, we multiply the equation by 5 and so we have

$$10(x - y)^2 + 15(y + \sqrt{5})^2 - 5z^2 = 75$$

and so we have

$$\frac{2}{25}(x - y)^2 + \frac{1}{5}(y + \sqrt{5})^2 - \frac{1}{25}z^2 = 1.$$

(If you can't see the change of coordinate directly you can set $\bar{x} = ax + by, \bar{y} = cx + d$) Therefore we see y is moved to $y + \sqrt{5}$ and thus we must also move x to $x + \sqrt{5}$ as we need to write $x - y$ as a linear form in terms of $\bar{x}, \bar{y}$. So the new origin is $(-\sqrt{5}, -\sqrt{5}, 0)$ and the new basis is $\{(1, -1, 0), (0, 1, 0), (0, 0, 1)\}$. The surface is an ellipsoid.

1,8B (a) (i),(ii) are book work. For (c), the first matrix sends standard basis to $\{(a,b),(c,d)\}$. Now we take the basis $\{(k,0),(0,1)\}$ then it sends $(k,0)$ to (ka,kb) which is $a(k,0)+bk(0,1)$ and it sends $(0,1)$ to (c,d) which is $c/k(k,0)+d(0,1)$ and so with respect to this basis the linear map has the second matrix so these two matrices represent the same linear map and so they are similar.

As similar is an equivalence relation so we may pick any basis we want to represent the matrix in a nice form. Suppose the matrix has eigenvalue a with multiplicity 2 then it has at least one eigenvector. Therefore, we take

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

for some b and $b=0$ we have the first case and if $b \neq 0$ by (a)(iii) we can assume $b=1$ by picking $k=b$.

For (c), by direct computation we have the following condition for r if B is orthogonal

$$\frac{1}{2}r^2+\frac{3}{4}=1, -\frac{1}{2}r^2+\frac{1}{4}=0, r^2+\frac{1}{2}=1$$

and hence we can pick $r=\pm\frac{\sqrt{2}}{2}$. The characteristic polynomial of $B(r_0)$ is

$$(t-1)(t^2-\sqrt{2}t+1)=0$$

and the only real eigenvalue is 1 and for $t=1$ the eigenvector is $(1,1,0)$. The images of $(1,-1,0)$ and $(0,0,1)$ are

$$(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -1), (\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2})$$

and by considering the dot product we conclude that B is a rotation (as it is rotation on the orthogonal basis) by θ where $\cos\theta=\frac{\sqrt{2}}{2}$.