PartIA Group

zc231

Each question will be labeled in the form $\alpha, \beta\gamma$ where $\alpha \in \{1,2,3,4\}$ represents the paper number, $\beta\gamma$ represents the question number in that paper. For example, 1,11G means question 11G in paper 1. I will omit the proofs in the notes or book work. The solutions provided might not be the best ways to solve the problems and if you find any mistakes or if you have any elegant ways of solving some of the problems please email me at zc231@cam.ac.uk.

2009

3,1D book work (any isometry which fixes the origin is represented by orthogonal matrix). For the last part consider the glide reflection (which is composition of a translation and a reflection) and it cannot be achieved by two reflections because a composition of two reflection is either a translation or rotation (as it has the form $v \mapsto Av + w$ where $\det A = 1$).

3,2D Consider A_4 and $k = 6$ or A_5 with $k = 30$ (A_5 is simple and if there is a subgroup of order 30 it must be normal).

3,5D Consider the action of symmetries on the set of vertices. Pick any vertex v. It is clear that the action is transitive so the orbit has size 6. Now if we fix v, then the vertex which is opposite to v is also fixed because the line joining them is mapped to some line joining opposite vertex. Then for one of the adjacent vertex u, we have 4 possible images and the image of u determines the image of the vertex opposite to u and then we have 2 possibilities for the last two vertices. Therefore the stabilizer has size 8 and so $|G| = 48$.

D has size 3 and so by Orbit-Stabilizer the stabilizer of any line has size 16. Let $D = \{d_1, d_2, d_3\}$. Suppose d_1 is fixed, and if in addition d_2 is fixed then d_3 is also fixed. We work out the stabilizer of d_1, say. If d_2 is not fixed and so $d_2 \mapsto d_3, d_3 \mapsto d_2$ and this is generated by a single rotation r and reflections which swaps vertices of d_i, s_i. If d_2 is fixed then we achieve these are generated by s_i. So if you are not clear how this works you can label the 6 vertices by $1,2,3,4,5,6$ and let d_1, d_2, d_3 be lines between $1,2$, $3,4$ and $5,6$ respectively. Then

$$r = (3546), s_1 = (12), s_2 = (34), s_3 = (56)$$

and then the stabilizer is generated by these elements. Note that $s_3 = r^2 s_2$ and s_1 commutes with every element in the group and we also have $s_2 r s_2^{-1} = r^{-1}$ so the subgroup generated by r, s_2 is D_8. Therefore we conclude that the group is isomorphic to $D_8 \times C_2$.

3,6D The first part is book work. For the second part let $\tau = (ab)$ and consider two cases. The first case is when a, b lies in the different cycles of σ, then we compute

$$(ab)(ac_0c_2 \cdots c_k)(bd_0d_2 \cdots d_r) = (bd_0 \cdots d_r ac_0 \cdots c_k), k, r \geq 0$$

and the second case is when a, b lies in the same cycle of σ then

$$(ab)(ac_0 \cdots c_k bd_1 \cdots d_r) = (ac_0 \cdots c_k)(bd_1 \cdots d_r), k, r \geq 0.$$

All the other cycles in σ are unchanged and therefore we conclude $l(\tau\sigma) = l(\sigma) \pm 1$.

3,7D It is clear (by definition) that for each w there exists a unique z such that the cross ratio is z. Then we prove any four distinct points are collinear or lie on the same circle if and only if the cross ratio is real.

$$[a_0, a_1, a_2, z] = \frac{z-a_1}{z-a_0}\frac{a_0-a_2}{a_1-a_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

where

$$r_1 = \left|\frac{z-a_1}{z-a_0}\right|, r_2 = \left|\frac{a_0-a_2}{a_1-a_2}\right|, \theta_1 = \arg\frac{z-a_1}{z-a_0}, \theta_2 = \arg\frac{a_0-a_2}{a_1-a_2}.$$

Therefore it is real if and only if $\theta_1 + \theta_2 = 0, \pi$ and the result follows by basic geometry (not sure whether you can assume basic geometry, even you draw a picture). The alternative way is to use the fact cross ratio is invariant under mobius map so send a_0, a_1, a_2 to $0, 1, \infty$ then let z' be the image of z so now $[0, 1, \infty, z'] = \frac{z'-1}{z'} \in \mathbb{R} \cup \{\infty\}$ and so $z' \in \mathbb{R}$ so the image z' forms the real line. As mobius map sends line/circle to line/circle the result follows.

For each z, let w be the conjugate of $[a_0, a_1, a_2, z]$ then there exists a unique $J(z)$ such that $[a_0, a_1, a_2, J(z)] = w$ so it is well-defined. We have

$$[a_0, a_1, a_2, J^2(z)] = \overline{[a_0, a_1, a_2, J(z)]} = [a_0, a_1, a_2, z]$$

and by uniqueness we conclude $J(z) = z$.

It is clear that $[a_0, a_1, a_2, \bar{z}] = \overline{[a_0, a_1, a_2, z]}$ and so by uniqueness $J(z) = \bar{z}$. Let l be the line or circle through a_0, a_1, a_2 and let ϕ be any mobius map which takes l to the real line. Then we have

$$\overline{[a_0, a_1, a_2, z]} = \overline{[\phi(a_0), \phi(a_1), \phi(a_2), \phi(z)]} = [\phi(a_0), \phi(a_1), \phi(a_2), J(\phi(z))]$$

and since $\phi(a_i) \in \mathbb{R}$ we conclude that $J(\phi(z)) = \overline{\phi(z)}$ (note this does not imply $J(z) = z$ because not every $u \in \mathbb{C}$ comes from $\phi^{-1}(z)$ for some z). We also have

$$\overline{[a_0, a_1, a_2, z]} = [a_0, a_1, a_2, J(z)] = [\phi(a_0), \phi(a_1), \phi(a_2), \phi(J(z))]$$

therefore by uniqueness of cross ratio we conclude that $\phi(J(z)) = J(\phi(z)) = \overline{\phi(z)}$ and so

$$J(z) = \phi^{-1}\overline{\phi(z)}.$$

This is independent of the points a_0, a_1, a_2 because the map ϕ only depends on l.

3,8D The first part is book work. Then as matrix multiplication is a group hence closed, and since determinant is multiplicative so $\mathrm{SL}_2(\mathbb{Z})$ is a group. Similarly $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ is a group (as $\mathbb{Z}/2\mathbb{Z}$ itself is a group). Then reduction modulo 2 map is a homomorphism because for any $a \in \mathbb{Z}$ if $[a]$ represents the equivalence class of integers congruent to a mod 2 then we have

$$[a] + [b] = [a+b], [a][b] = [ab]$$

therefore the reduction map is a homomorphism. It is clearly surjective so the image is $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$. The size of the group is 6 because if you consider any matrix with entry $a, b, c, d \in \{0, 1\}$ then $a = 1$ then b, c can be 0 or 1 and d is determined by the discriminant condition. If $a = 0$ then $b, c = 1$ and $d = 0$ or 1 so we have 6 of them. Finally, as

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

so it is not abelian and hence it is isomorphic to S_3.

2010

3,1D book work.

3,2D (21)(34) the conjugacy class contains every element of cycle type $2-2$.

3,5D The stabilizer contains 4 element (rotate about the face by $90, 180, 270, 360$ degrees) and since the action of G on X is transitive so by Orbit-Stabilizer we have $|G| = 24$. The action is coordinate-wise, and has three orbits O_1, O_2, O_3 where $O_1 = \{(x, x) : x \in X\}$, $O_2 = \{(x, y) : y \mathrm{adjacentto} x\}$ and $O_3 = \{(x, y) : y \mathrm{oppositeto} x\}$ where O_1, O_3 both have size 6 and O_2 has size 24. It is clear that O_1, O_3 form an orbit. For O_2, if we are given (x, y) with y adjacent to x, then the image of y is adjacent to the image of x under any rotation.

N is normal if $gNg^{-1} = N$ for all $g \in G$. The stabilizer of any element is normal and so pick any $x \in X$ then the stabilizer of x has order 4 so it is normal.

3,6D Pick any $g \neq 1$ and so the order of g divides p, but as p is prime so g has order p and so g generates the group. Let G be abelian of order p^2. Then pick any $g \neq 1$. Suppose g has order p^2 then $G \cong C_{p^2}$ and if g has order p. If not, then search for another element $h \neq 1$ and if any h has order p^2 we have the same conclusion. Therefore we can assume every element, which is not identity, has order p (as the order of element is $1, p$ or p^2). Then pick g, h such that g is not in $\langle h \rangle$ and since G is abelian it is clear that each element in G has the form $g^i h^j, 0 \leq i, j \leq p-1$. Therefore it is isomorphic to $C_p \times C_p$.

A_4 has no subgroup of order 3 (if you have an element of cycle type $2-2$ and a 3-cycle they generate A_4) but D_{12} has a subgroup of order 6 so they are not isomorphic (there are lots of way to show they are not isomorphic).

3,7D Pick h with $h^{-1}x = y$ which is possible as the action is transitive and since $gy = y$ we have $gh^{-1}x = h^{-1}x$ which gives $hgh^{-1}x = x$ so $hgh^{-1} \in B$.

G_∞ consists of matrices with $c = 0$. For the set of fixed points, suppose $c = 0$, then $d \neq 0$ and g fixes ∞ if $a = d = \pm 1$, and g fixes ∞ and $b/(\frac{1}{a} - a)$ if $a \neq \pm 1$. Suppose $c \neq 0$, then we solve the quadratic equation

$$cx^2 + (d-a)x - b = 0.$$

If the discriminant is 0, which corresponds to $d^2 + a^2 - 2ad + 4bc = 0$ and one can replace bc by $ad - 1$ as the determinant of g is 1 so we have $(a+d)^2 = 4$ which gives $a + d = \pm 2$ (as you should be familiar by book work this corresponds to matrix with trace 2), and then the only fixed point is $\frac{a-d}{2c}$ (where you can replace d by $\pm 2 - a$). Finally, if $c \neq 0$ and the trace of g is not ± 2, then we have two fixed points corresponding to the roots of the quadratic polynomials.

For the last part, as the action is transitive so pick h with $h^{-1}\infty = x$ then $hgh^{-1} \in B$ by exactly the same argument as in the second part.

3,8D Conjugation is clearly an action on element and it maps proper subgroup to proper subgroup so it defines an action on X (you probably need to write down something like $(g^{-1}h_1g)(g^{-1}h_2g) = g^{-1}h_1h_2g$).

Let H be the stabilizer of B then $h \in H$ if and only if $hBh^{-1} = B$. Clearly if $h \in B$ then $h \in H$ and so $|H| \geq |B|$. Then by Orbit-Stabilizer we conclude that the size of orbit is at most $[G : B]$. Suppose each $g \in G$ is conjugate to an element of G then this means for each $g \in G$, there exists $b \in B$ and $h \in G$ such that $g = hbh^{-1}$ and so $g \in hBh^{-1}$. Therefore, the union of the orbits B contains every element in G. We then count the number of elements of the union of orbits of B. By previous part the orbit has at most $[G : B]$ groups and each has the same size as B. But each group contains 1 so 1 is counted $[G : B]$ times and so the size of the union is strictly less than $[G : B]|B| = |G|$, which gives a contradiction.

2011

3,1D Stabilizer has size 4 and orbit has size 12 so G has order 48. The orbit of (x, x) is $\{(x, x) : x = 1, \ldots, n\}$ and the orbit of (x, y),$x \neq y$ is $\{(x, y) : x \neq y, x = 1, \ldots, n, y = 1, \ldots, n\}$ because for any $x_1 \neq y_1$, pick g with $gx = x_1, gy = y_1$

3,2D If k is odd then $k\big|n$ and let $m = \frac{n}{k}$ then $\langle \alpha^m \rangle$ has order k. If k is even let $k = 2k'$ so $k'\big|n$ and let $m = \frac{n'}{k}$ then $\langle \alpha^m, \beta \rangle$ has order k (notation: $\alpha^n = \beta^2 = 1$).

3,5D (a),(b) are book work. For (b), if g has cycle type $n_1 - n_2 - \cdots - n_r$ then the order of g is the least common multiple of $n_i, i = 1, \ldots, r$. (c) is book work.

S_n is naturally a subgroup of S_{n+2} and let S be the image of S_n in S_{n+2}. We now establish an isomorphism between S and A_{n+2}. Define a map $\phi : S \to A_{n+2}$ as follows: if g is an even permutation, $\phi(g) = g$. If g is an odd permutation, $\phi(g) = g \cdot (n + 1 n + 2)$ where $(n + 1 n + 2)$ is a transposition. Then ϕ is a well-defined map from S to A_{n+2}. Suppose we have elements g_1, g_2. If g_1, g_2 both even then $\phi(g_1 g_2) = g_1 g_2$. If g_1 is even, g_2 is odd then $\phi(g_1 g_2) = g_1 g_2 (n + 1 n + 2) = \phi(g_1)\phi(g_2)$ and if g_1 is odd, g_2 is even, then $\phi(g_1 g_2) = g_1 g_2 (n + 1 n + 2) = g_1 (n + 1 n + 2) g_2 = \phi(g_1)\phi(g_2)$. If g_1, g_2 both odd then $\phi(g_1 g_2) = g_1 (n + 1 n + 2) g_2 (n + 1 n + 2) = g_1 g_2 = \phi(g_1)\phi(g_2)$ so it is a homomorphism. It is clearly injective and so this identifies S with a subgroup of A_{n+2}.

3,6D (a) is direct computation. The second part (a) follows by constructing the reduction modulo 5 map, and it is easy to check it is a homomorphism and the group in the question is the kernel of the map.

We first compute the size of $\mathrm{SL}_2(\mathbb{F}_5)$. Consider any matrix with entries a, b, c, d and $ad - bc = 1$. If $a \neq 0$, then we are free to pick b, c to be anything and d is determined by $ad - bc = 1$ so we have 100 matrices. If $a = 0$, then d can be anything and $b, c \neq 0$, once one of b, c is picked the other one is determined by $ad - bc = 1$ so we have 20 matrices and so in total we have 120. As special linear group is the quotient of general linear group by determinant and $\mathbb{F}_5^*$ has 4 element so we conclude $\mathrm{GL}_2(\mathbb{F}_5)$ has size 480. Ah well it seems that I did the second part first but I think it is easier to think in this easy.

$\mathrm{SL}_2(\mathbb{F}_5)$ has index 4 in $\mathrm{GL}_2(\mathbb{F}_5)$ and an index 2 subgroup can be a lift of $\mathrm{SL}_2(\mathbb{F}_5)$ up to some scaler. For example, take the group consisting of elements which are either in $\mathrm{SL}_2(\mathbb{F}_5)$ or $2\mathrm{SL}_2(\mathbb{F}_5)$ (they are disjoint as the later has determinant 4).

3,7D (a) is book work. By Orbit-Stabilizer $|Y||\mathrm{stab}(x)| = p^n$ for some $x \in Y$ and so either $|Y| = 1$ or $p\big||Y|$. Let G acts on itself by conjugation, and let c_i be the number of orbits of size i, then

$$p^n = |G| = c_1 + pc_p + p^2 c_{p^2} + \cdots$$

and so $p\big|c_1$, in particular $c_1 > 1$. Suppose $Z(G)$ has order p^{n-1} and let $x \notin Z(G)$. Then the quotient group $G/Z(G)$ contains $xZ(G)$ and the order of $G/Z(G)$ is p so it is cyclic. Therefore the quotient is generated by $xZ(G)$ and hence every element in G can be written as $x^i z, i \leq p - 1, z \in Z(G)$. Then take two elements $g = x^i z_1, h = x^j z_2$ we have

$$gh = x^{i+j} z_1 z_2 = hg$$

which implies that G is abelian and so $Z(G) = G$, which is a contradiction.

3,8D For (a), let $g \notin H$ then the coset gH must be the same as Hg because $gH \cup H = G = Hg \cup H$. Therefore, $gHg^{-1} = H$.

Then write $D_{2n} = \langle \alpha, \beta \rangle$ where $\alpha^n = \beta^2 = 1$ the subgroup $\langle \alpha \rangle$ is normal as the size is half of $|D_{2n}|$. Then take $k = 3$ and $G = S_3, H = C_2$ generated by a transposition, then H is not normal.

Suppose K is a normal subgroup of A_5. As $gkg^{-1} \in K$ for all $g \in G, k \in K$ so K is a union of conjugacy classes and it must contain 1. Then you work out the conjugacy classes of A_5 (which is standard book work but you have to remember how to do that) and use Lagrange's theorem to conclude that K cannot be a union of conjugacy classes containing 1.

2012

3,1E Use Lagrange's theorem on the subgroup generated by each element.

Let $g, h \in G$. Then $gh \in G$ so $(gh)^2 = 1$ and so $ghgh = ghg^{-1}h^{-1} = 1$ because $g^2 = h^2 = 1$. So $gh = hg$. Example: D_8.

3,2E book work. No, (123) is not conjugate to (132).

3,5E (i) is book work. The homomorphism is injective if and only if for all $g_1 \neq g_2$, there exists $x \in X$ such that $g_1 x \neq g_2 x$. Write $H = \text{stab}(x)$. Let $g \in G$ and then each element in gHg^{-1} fixes gx. Conversely if k fixes gx then $g^{-1}kg \in H$ and so $k \in gHg^{-1}$. This shows that the stabilizer of gx is gHg^{-1}. Therefore, the intersection of all conjugates of H consists of element which fixes gx for all $g \in G$ and we have a single orbit so we conclude that the intersection of H consists of element which fixes every element. Therefore, if the intersection is non-trivial then we have an element $g \neq e$ such that $gx = x = ex$ for all x and so the homomorphism is not injective. Conversely, if we have $g_1 \neq g_2$ but $g_1 x = g_2 x$ for all x then $g_2^{-1} g_1 \neq e$ is an element which fixes all x and hence lies in the intersection of the conjugates of H.

Suppose Q_8 is isomorphic to some subgroup of $S_n, n \leq 7$ then this gives an injective homomorphism from Q_8 to S_n which gives an action of Q_8 on $X = \{1, \ldots, n\}$. For each $x \in X$, the orbit of x has at most n elements ($n \leq 7$) so by Orbit-Stabilizer we conclude that the stabilizer is non-trivial for each x. But every subgroup of Q_8 contains -1 and so -1 must be in the stabilizer of every x. Therefore -1 fixes every element so the homomorphism $Q_8 \to S_n$ cannot be injective.

3,6E Identify each element in G by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The orbit of 0 consists every real number and ∞, and the stabilizer consists of matrices with $b = 0$. The orbit of i contains every non-real number with positive imaginary part, and the stabilizer consists of matrices with $a = d, b = -c$ and $a^2 + b^2 = 1$ (so basically you can parameterize this by $a = \cos\theta, b = -\sin\theta$). The orbit of $-i$ contains every non-real number with negative imaginary part, and the stabilizer is the same as the stabilizer of i. Thus G has exactly three orbits.

The orbit contains element of the form $a^2 i + ab$ and so it contains non-real elements with positive imaginary part. Since H and G has the same orbit of i so for each $g \in G$, there exists $h \in H$ such that $gi = hi$ and so $h^{-1}g$ lies in the stabilizer of i, which we have described above, has the form k as in the question.

Take elements k_i with angle θ_i, $i = 1, 2$ we have $k_1 k_2 = k_3$ with angle $\theta_3 = \theta_1 + \theta_2$ so the collection of these elements form a group. Suppose now $h_1 k_1 = h_2 k_2$ then $k_2 k_1^{-1} = h_2^{-1} h_1 \in H$, and so we must have $\sin(\theta_2 - \theta_1) = 0$. So $\theta_2 = \theta_1$ or $\theta_2 = \theta_1 + \pi$ if $\theta_2 = \theta_1$ we have $k_2 = k_1$, and if $\theta_2 = \theta_1 + \pi$ we have $k_2 = -Ik_1$ and so $h_2 = -Ih_1$ so we have two ways.

3,7E The first part is clear as the discriminant is multiplicative. The second part is direct computation. $H = \langle g \rangle$ act on $\mathbb{F}_p \times \mathbb{F}_p$. For any (x, y), by Orbit-Stabilizer theorem, the size of the orbit under H is either 1 or p. We look for some $(x, y) \neq (0,0)$ such that the size of the orbit is 1 (so it is fixed by H). Clearly the orbit of $(0,0)$ has size 1. Then

$$p^2 = |\mathbb{F}_p \times \mathbb{F}_p| = c_1 + pc_p$$

where c_1 is the number of orbits of size 1 and c_p is the number of orbit of size p. So $p \big| c_1$ and so $c_1 > 1$.

Take (x, y) fixed by g and there exists r, s such that $xr - sy = 1$ in $\mathbb{F}_p$ (as x, y not both 0) we have

$$g \begin{pmatrix} x & s \\ y & r \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & s \\ y & r \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} x & s \\ y & r \end{pmatrix}^{-1} g \begin{pmatrix} x & s \\ y & r \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and so g is conjugate to a stabilizer of $(1, 0)$. But every stabilizer of $(1, 0)$ must have $a = 1, c = 0$. Further as g has order p so $(\det g)^p = 1$ but $(\det g)^p = (\det g)$ so we conclude $d = 0$ and therefore g is conjugate to

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

for some b and now we show this matrix is conjugate to the required form. We have

$$\begin{pmatrix} 1 & 1 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & b^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

3,8E To check it defines a group operation, it is clear the operation is closed and associativity is checked by direct computation. The group has identity $(0, 0)$ and the inverse of (x, u) is $(-x, -a^{-x}u)$. G is abelian if and only if

$$a^y u + v = a^x v + u, \forall x, y, u, v.$$

Therefore, we require $a^x = a^y = 1$ for all x, y and so $a = 1$.

The map $G \to \mathbb{Z}/(p-1)\mathbb{Z}, (x, u) \mapsto x$ defines a homomorphism (by using $z = x + y$) and the kernel is K so K is a normal subgroup. Suppose $a = 1$ then $G \to \mathbb{Z}/p\mathbb{Z}, (x, u) \mapsto u$ is a homomorphism with kernel H so H is normal. Conversely, take a general element (y, v) then we compute

$$(y, v) * (x, 0) * (-y, -a^{-y}v) = (y, v) * (x - y, -a^{-y}v) = (x, a^{x-y}v - a^{-y}v)$$

and if H is normal then we require $a^{x-y} = a^{-y}$ for all x, y and hence $a = 1$.

The last part is exactly how I proved K is normal above.

2013

3,1D Let $\phi : G \to H \times K$ be a map $\phi(g) = (h, k)$. We check ϕ is well-defined. Suppose $h_1k_1 = h_2k_2$ then $h_2^{-1}h_1 = k_2k_1^{-1}$ and so they lie in $H \cap K$, which is a subgroup of both H and K and so the order divides the greatest common divisor of $|H|$ and $|K|$ by Lagrange. By (i) $H \cap K = \{1\}$ and so $h_1 = h_2, k_1 = k_2$. Then for any $h \in H, k \in K$ we have

$$hkh^{-1}k^{-1} \in K \cap H = \{1\}$$

because H, K are normal and so $hk = kh$. Therefore, ϕ is a homomorphism. It is clearly surjective by (ii) and it is injective by construction.

3,2D $C_2 \times C_2$ is abelian but not cyclic. Fix a generator c of C_n and let g be its image in G ($f : C_n \to G$) then $g^n = (f(c))^n = f(c^n) = 1$. The images of any element c^k is g^k.
Fix a generator c. You can send g to any element g of cycle type 4,2 − 2 or 1.

3,5D (a) is book work (Cayley's theorem) with $X = G$. H acts transitively on X (as rotation acts transitively). If one of the edge is fixed then any permutation must take an adjacent edge to another adjacent edge and so we have 4 elements (maybe you should draw a picture to help). Hence by orbit-stabilizer theorem we have $|H| = 48$.

The action defines a homomorphism (book work) and $\mathrm{Sym}(X)$ has order 12! so the number of cosets is 12!/48. It is not normal. Consider an element of rotation, say given by

$$(1234)(5678)(9101112), 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \in X$$

and we know two elements are conjugate to each other if and only if they have the same cycle type. So clearly we have more than 48 elements of this cycle type (so any 4-cycle has 6 elements with the same cycle type and so we have at least 6^3 of them).

3,6D If you think of $\mathbb{F}_p \cup \{\infty\}$ as projective line then the action is given by $\frac{ax+b}{cx+d}$ from a matrix with entries a, b, c, d and $x \in \mathbb{F}_p$, and the image of ∞ is $\frac{a}{c}$. The orbit of ∞ is $\mathbb{F}_p \cup \{\infty\}$ and the stabilizer is the matrix with entry $c = 0$. Then we need $ad = 1$ and so we have $p - 1$ choices whereas we can have b being any element in $\mathbb{F}_p$. Therefore by orbit-stabilizer the order of the group is $(p + 1)p(p - 1)$.

It is clear that the only fixed point of A is ∞ and suppose there exists g with $g^{-1}Ag = B$ then as B also fixes ∞ so we conclude $g\infty$ is a fixed point of A. Hence $g\infty = \infty$ so we only need to search g in the stabilizer of ∞. Thus, for a general element

$$g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

we have

$$gAg^{-1} = \begin{pmatrix} 1 & a^2 \\ 0 & 1 \end{pmatrix}.$$

Therefore such g exists if and only if 3 is a square mod p. If $p = 11$, then $6^2 \equiv 3 \bmod 11$.

Clearly G consists of invertible matrices and it is easy to check G is a group. Then consider the map

$$f : G \to \mathbb{R}, g \to a$$

then clearly (by direct computation) f is a surjective homomorphism with kernel H so H is normal and $G/H \cong \mathbb{R}$.

Let $g_1, g_2 \in G$ with entries a_i, b_i, x_i, $i = 1,2$ and $g_1g_2 = g_2g_1$ if and only if $a_1b_2 = a_2b_1$. Now fix $g_1 \in Z(G)$ and vary $g_2 \in G$, so we look for a_1, b_1 such that $a_1b_2 = a_2b_1$ for all a_2, b_2. Let $a_2 = 0, b_2 \neq 0$ so we have $a_1 = 0$ and similarly $b_1 = 0$. Thus, the center $Z(G)$ consists of elements with $a, b = 0$. Consider the map

$$f : G \to \mathbb{R}^2, g \to (a, b)$$

then by direct computation one checks f is a surjective homomorphism with kernel $Z(G)$ so the quotient is $\mathbb{R}^2$.

3,7D If $G = \langle \alpha, \beta \rangle$ then the elements of order 2 are $\alpha^k\beta$, $1 \leq k \leq 2n$. We have two conjugacy classes, one containing $\alpha^k\beta$ with k even and the other one contains the elements with k odd.

(b)(i): One direction is clear, if $H \subset K$ or $K \subset H$ then the union is just one of them and so it is a group. Suppose now H contains an element h not in K and K contains an element k not in H then we check hk is not in $H \cup K$. Suppose $hk \in H \cup K$ then $hk \in H$ or $hk \in K$, which then shows $k \in H$ or $h \in K$.

For (ii) consider the set $K \cup H$. For all $g \in G$, either $g \in H$ or $g \in G \backslash H$, in which case $g \in K$ and so $K \cup H = G$ is a group. Then by (i) we conclude either $H \subset K$ or $K \subset H$. But K contains some element not in H so $H \subset K$ and so

$$G = K \cup H = K.$$

3,8D The center is a union of 1-conjugay classes. Consider the action of G on itself by conjugation. Then by orbit-stabilizer theorem we conclude that the size of orbit of each element is a power of p (in this case the orbit of an element is just the conjugacy class). Finally, G is the union of all conjugacy classes and therefore the center is non-trivial because if c_i denote the number of conjugay classes of size i then

$$|G| = c_1 + pc_p + p^2c_{p^2} + \cdots$$

and so $p \big| c_1$.

For the second part, we know there is a non-trivial center of G and so the center has size p or p^2. In both case we conclude that there is a subgroup of order p which commutes with any other elements, say H, and so H is normal and we form the quotient group G/H, which has order p and this must be cyclic, generated by some element, say xH. Also as $|H| = p$ so it is also cyclic say generated by some element y. Therefore, each element in G can be represented by x^iy^j and as H commutes with any element so we have

$$x^{i_1}y^{j_1}x^{i_2}y^{j_2} = x^{i_2}y^{j_2}x^{i_1}y^{j_1} = x^{i_1+i_2}y^{j_1+j_2}$$

and therefore G is abelian. Finally, if x has order p then $G \cong C_p \times C_p$ and x has order p^2 then $G \cong C_{p^2}$.