PartIB Groups Rings and Modules

zc231

Each question will be labeled in the form $\alpha, \beta\gamma$ where $\alpha \in \{1, 2, 3, 4\}$ represents the paper number, $\beta\gamma$ represents the question number in that paper. For example, 1,11G means question 11G in paper 1. I will omit the proofs in the notes or book work. The solutions provided might not be the best ways to solve the problems and if you find any mistakes or if you have any elegant ways of solving some of the problems please email me at zc231@cam.ac.uk.

2009

2,2F $56 = 2^3 7$ so $n_2 \equiv 1 \bmod 2$ and $n_2 \big| 7$, $n_7 \equiv 1 \bmod 7$ and $n_7 \big| 8$. If both $n_2, n_7 > 1$ then $n_2 = 7$ and $n_7 = 8$. Each group of order 7 is cyclic and so the Sylow-7 groups are pairwise disjoint apart from the identity and we so we have 48 elements from the union of Sylow-7 groups and $n_2 = 7$ implies we have at least $7 + 6$ elements and then we have 61 elements which is a contradiction.

3,1F Take any ideal I and if $I \neq 0$ pick any element f with the least degree in I and assume f is monic because F is a field so we can multiply f by the inverse of its leading coefficient. Now for any $g \in I$, by assumption the degree of g is at least the degree of f and since f is monic there exists q, r such that $g = fq + r$ with $r = 0$ or $\deg r < \deg f$ (this is not Euclidean algorithm, and this can always be done if f is monic) and so $g \in \langle f \rangle$. For the second part, take $\langle x, y \rangle$.

4,2F Let N be the collection of torsion elements. Then it is a submodule because if $n_1, n_2 \in N$, so we have $r_1, r_2 \neq 0$ with $r_1 n_1, r_2 n_2 = 0$ then $r_1 r_2 (n_1 + n_2) = 0$ (as R is a domain so $r_1 r_2 \neq 0$) and for any $r \in R, n \in N$ it is clear that $rn \in N$. M/N is torsion free because if $a \in M$ is torsion then $a \in N$ and the representative of a in M/N is 0. This is unique because if m is torsion but $n \notin N$, then $m \neq 0$ in M/N and so M/N is not torsion-free (so with property (i) and (ii) the only one is the set of torsion free elements).

1,10F $4 = (1 - \sqrt{-3})(1 + \sqrt{3}) = 2 \cdot 2$ and so the factorization is not unique (to show $1 \pm \sqrt{-3}$ and 2 are irreducibles, consider the norm, which is $x^2 + 3y^2$ for any $x + \sqrt{-3}y$). Let $R = \mathbb{Z}[\omega]$ where $\omega = \frac{-1+\sqrt{-3}}{2}$. Then the quotient of the additive group $R/\mathbb{Z}[\sqrt{-3}]$ has order 2 and has representatives $0, \omega$ because $2\omega \in \mathbb{Z}[\sqrt{-3}]$ and $2\mathbb{Z}[\sqrt{-3}] \subset R$. Finally, R is a principal ideal domain. In fact it is a Euclidean domain, and it is standard lattice argument (which I believe you have seen in the lecture) to show the norm function is Euclidean.

2,11F The first part is book work. If $G/Z(G)$ is cyclic, pick a generator say x, and so each $g \in G$ can be written as $x^i z, z \in Z(G)$ and so for any i, j and $z_1, z_2 \in Z(G)$ we have

$$x^i z_1 x^j z_2 = x^j z_2 x^i z_1 = x^{i+j} z_1 z_2$$

and so G is abelian so we have $Z(G) = G$. Then suppose $|G| = p^3$ and $Z(G)$ has order p^2 we see $G/Z(G)$ is cyclic and so $Z(G)$ should have order p^3 (and clearly $Z(G)$ cannot have order p^3 as G is abelian).

G has order p^3 and it is clear G is not abelian and so the center has order p. By direct computation one checks that the matrices in G with $a, c = 0$ commute with any other matrices and they form a subgroup of order p, which implies it is the center.

3,11F Suppose $ab \in I$ but $a, b \notin I$ then the ideal $\langle a \rangle + I$ contains I and if $a \notin S$ then $\langle a \rangle + I$ is disjoint from S which is a contradiction and so $a \in S$. Similarly $b \in S$ and so $ab \in S$, which then implies $ab \in S \cap I$, which is a contradiction. For the second part take the field of fractions.

For the last part take T maximal among ideals disjoint from S and so T is prime. Therefore, R/T is an integral domain, and so we have a ring homomorphism g from R/T to the field of fractions of R/T, say F. Let f be the quotient morphism $R \to R/T$ and so $gf : R \to F$ is a homomorphism. For any $x \in I$ we have $x \in T$ and so $f(x) = 0$ and so $gf(x) = g(0) = 0$. For $y \in S$ we have $y \notin T$ so $f(y) \neq 0$ and so $gf(y) \neq 0$ by construction of field of fractions.

4,11F The first part is book work. Suppose P is free with a basis E. Let $f : M \to N$ be a surjective homomorphism and g any homomorphism from P to N. As f is surjective, for each $e \in E$, let $t_e \in M$ with $f(t_e) = g(e)$ and define $h(e) = t_e$. Then for each $p \in P$, there exists $r_i \in R$ such that, $p = r_1 e_1 + \cdots + r_k e_k, e_i \in E$ and so h is determined by the $h(E) = \{h(e) : e \in E\}$ which extends to a homomorphism. Since $fh(e) = g(e)$ for all $e \in E$ so $fh(p) = g(p)$ for all $p \in P$ and so P is projective.

Let P be a finitely generated projective module with a basis $\{e_1, \ldots, e_k\}$. Let $M = R^k$ be the free module generated by $\{e_1, \ldots, e_k\}$ (don't confuse this with P) and let $N = P$. Let $f : M \to P$ by $f(0, \ldots, 1, \ldots, 0) = e_i$ for 1 on the i-th position and extend to a homomorphism. Then clearly f is a surjection. Let $g : P \to P$ be identity map. Then as P is projective there exists $h : P \to M$ such that $fh = g$. As g is identity so h is injective and so P can be identified with a submodule of M. Since M is a finitely generated free module over a principal ideal domain, so any submodule is free.

2010

2,2H Each conjugate class contains elements of the same cycle type.

3,1H $A = \mathbb{Z}$. Let $I = 12\mathbb{Z}$ then $A/I = \mathbb{Z}/12\mathbb{Z}$. $4^2 \equiv 4 \bmod 12$ so 4 is idempotent and 6 is nilpotent as $6^2 \equiv 0 \bmod 12$. Suppose $A = \mathbb{C}[X]$. Take $I = \langle X^2(X-1)\rangle$. Then X^2 is idempotent because $X^4 \equiv X^3 \bmod I$ and $X^3 \equiv X^2 \bmod I$. The element $X(X-1)$ is nilpotent.

4,2H Suppose $(a,b) = d > 1$ then $d(\frac{a}{d}e_1 + \frac{b}{d}e_2) = 0 \in M/N$ but $d \neq 0$ and $\frac{a}{d}e_1 + \frac{b}{d}e_2 \neq 0$ in M/N. Conversely, let $(a,b) = 1$ and so if $r_1e_1 + r_2e_2 \in N$ then we have some integer k with $r_1 = ak, r_2 = bk$. Suppose M/N is not free then $r_1e_1 + r_2e_2 = kae_1 + kbe_2$ can be written as $m(s_1e_1 + s_2e_2)$ for $0 < |s_1| < a, 0 < |s_2| < b$ but as $(a,b) = 1$ this cannot happen.

1,10H The first part is book work. If $D_8 = \langle \alpha, \beta\rangle$ with $\alpha^4 = \beta^2 = 1$ and $\beta\alpha\beta = \alpha^3$, then $\langle\beta\rangle$ is not a normal subgroup of order 2. But clearly $H = \langle \beta, \alpha^2\rangle$ is normal because it has index 2 and $\langle\beta\rangle$ is a normal subgroup of H which is not normal in D_8.

2,11H (i) Suppose not let $i \in I$ and $j \in J$ with $i, j \notin P$, and as P is prime we have $ij \notin P$, but $ij \in IJ \subset IJ$ and this is a contradiction.

(ii) Let $R = \mathbb{Z}$ and let $I = J = 2\mathbb{Z}$ but $IJ = 4\mathbb{Z}$ and $I \cap J = I = 2\mathbb{Z}$.

(iii) Prime ideals in R/A can be identified with prime ideals containing A. Let P contain IJ, then P contains I or J and hence P contains $I \cap J$. Conversely, if P contains $I \cap J$, then as $IJ \subset I \cap J$ so P contains IJ. Therefore, a prime ideal P contains IJ if and only if P contains $I \cap J$.

3,11H Take any $x \in S$, if x is irreducible we are done, and if not let $x = x_1y_1, x_1, y_1 \in S$. Suppose both x_1, y_1 are irreducibles then we are done, if not, without loss of generality let $x_1 = x_2y_2$. Repeat the above argument so either this terminates which then shows x is a product of finitely many irreducibles, or we obtain a sequence $x_1, x_2, x_3, \ldots$ with the property that $x_{i+1}\big|x_i$ and so $I_i = \langle x_i\rangle \subset \langle x_{i+1}\rangle = I_{i+1}$ for each i and I_i is strictly contained in I_{i+1} because $y_i \neq 0$ or any unit in R. This gives a strictly ascending chain which contradicts R being Noetherian.

4,11H Take generators, $e_1, e_2, e_1 + e_2, e_1 + 2e_2$ and they give different 1-dimensional space. Stabilizer of $\langle e_1\rangle$ consists of matrices with $c = 0$. If $c = 0$ then $a, d \neq 0$ as $ad - bc \neq 0$ so we have two choices for each a and d and we have 3 choices for b so $|K| = 12$. By direct computation, the action is transitive so by orbit-stabilizer theorem $|G| = 48$.

By considering the intersection of stabilizers of $\langle e_1\rangle, \langle e_2\rangle$ we conclude that H is the set of scaler matrices and $|H| = 2$. Fix a bijection $X \to \{1,2,3,4\}$. Define a map from $G \to S_4$ by sending g to the permutation which corresponds to the action of g on X. Then it is clear this is a homomorphism. The kernel is H and so G/H is isomorphic to a subgroup of S_4 and in fact this is an isomorphism as $|G/H| = |S_4| = 24$.

2010

2,2F The only element of order 2 in Q_8 is -1 but we have many elements of order 2 in D_8.

3,1F For each $a \neq 0$, define $f_a : A \to A$ by $f_a(x) = ax$. This is clearly a linear map and it is injective as A is an integral domain. Then by rank-nullity this is surjective and so f is surjective so there exists $b \in A$ such that $ab = 1$.

4,2F Let $I_n = \langle 2^n \rangle$.

1,10F (i) is book work. Let G be a group of order pq. Then $n_p \big| q$, $n_p \equiv 1 \bmod p$, $n_q \big| p$, $n_q \equiv 1 \bmod q$. Suppose $n_p > 1$ then $n_p = q$ and each group of order p is cyclic of prime order and so the Sylow-p subgroups of G are disjoint apart from the identity 1 and so the union gives $1+(p-1)q = pq-q+1$ elements. Therefore $n_q = 1$ otherwise the number of element exceeds pq.

2,11F Use the norm function to show $\mathbb{Z}[i]$ is Euclidean (which is standard) and as it is Euclidean, prime element is equivalent to irreducible element (you probably need to prove this, given this question is pretty short) and $4 + i$ has norm 17 so it must be irreducible as 17 is irreducible.

3,11F The smith form is $2I$. Then by structure theorem $M \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

4,11F The first part is book work. No, for example, the polynomial ring with infinitely many generators

$$K[x_1, x_2, x_3, \ldots]$$

2011

3,1I By considering the norm function on $\mathbb{Z}[\sqrt{-3}]$, 2 is irreducible (as no element has norm 2). 2 is not prime for example $2 \big| 4 = (1 + \sqrt{-3})(1 - \sqrt{-3})$ but 2 divides neither of them.

2,2G Let X be the set of opposite edges (each edge has exactly one edge which is not adjacent to it). The group of symmetries act on X and it is transitive so we obtain a surjective homomorphism (for example consider the action of reflection which swaps two faces and this gives a transposition in S_3).

4,2G If $1+rX$ is idempotent then we need $2r = r, r^2 = 0$ and so we have $r = 0$. Suppose $(1+rX)^k = 0$ then we have $1 = 0$ (and this implies R is a zero ring so $r = 0$). Let $R = \mathbb{Z}/12\mathbb{Z}$ and let $r = 6$ then $(1 + 6X)^3 = 1 + 6X$.

1,10G The first part is book work. Suppose G is simple and $G \neq H$, write $|G| = p^r m$ where $(m, p) = 1$ so $n_p \big| m$ (as it is prime to p and divides $|G|$). As G is simple so $n_p > 1$, and as Sylow-p subgroups are conjugate to each other so G acts on the set of Sylow-p subgroups by conjugation and the action is transitive. If $g \in G$ has trivial action then as $n_p > 1$, we have g lies in the intersection of the normalizers of all Sylow-p subgroups. Then we know the intersection of all normalizers of Sylow-p subgroups is normal (check it by definition) and since G is simple we must have $g = 1$. Therefore this shows G injects into a subgroup of S_{n_p} and so $|G| \big| n_p!$.

For the last part, the only prime factors are $2, 5$ and $n_2 \equiv 1 \bmod 2$, $n_2 \big| 5^6$, and $n_5 \equiv 1 \bmod 5$, $n_5 \big| 2^6$. Suppose it is simple then $n_5 > 1$, but $n_5 \equiv 1 \bmod 5$ so $n_5 = 16$ and so we cannot have $|G| \big| n_5!$ because the power of 5 is at most 5^3 from $5, 10, 15$.

2,11G Replace X by $X + 1$ we have

$$1+(1+3X+3X^2+X^3)+(1+6X+15X^2+20X^3+15X^4+6X^5+X^6) = 3+9X+18X^2+21X^3+15X^4+6X^5+X^6$$

which is 3-Eisenstein. For (ii), if it has a linear factor over $\mathbb{Z}_2[X]$ then it has a root (which is 0 or 1), but evaluating at 0 or 1 gives 1. Therefore if it is reducible over $\mathbb{Z}_2[X]$ it is a product of a degree 2 irreducible factor and a degree 3 irreducible factor. The only degree two irreducible factor is $1 + X + X^2$ by Euclidean algorithm (or division) we see $1 - X^2 + X^5$ is not divisible by $1 + X + X^2$.

3,11G $\mathbb{Z}_2[X]$ is principal ideal domain and so each irreducible polynomial is maximal (maybe you need to explain why this is true in the exam) and $1 + X + X^2$ is irreducible so the quotient is a field. For the second one, $X^2 + X + 1 = (1 - X)^2$ in $\mathbb{Z}_3[X]$ and it is not an integral domain hence not field. For (iii), suppose we only have irreducibles with degree less than M, say $f_1, \ldots, f_n$. Then let $f = f_1 f_2 \cdots f_n + 1$, and if $f_i \big| f$ then $f_i \big| 1$ which can not happen and so f is not divisible by any f_i so f must be divisible by some irreducibles other than f_i.

For (iv), let $f(x) = g(x)h(x)$ and $g(x) = f(x)k(x)$. If $f(x) = 0$ then $g(x) = 0$ and if $f(x) \neq 0$ then $g(x)h(x) \neq 0$ and so $g(x) \neq 0$ so $f(x) = 0$ if and only if $g(x) = 0$. We show $h(x)$ is invertible with inverse $k(x)$. For $f(x), g(x) \neq 0$ clearly $h(x)k(x) = 1$. Suppose $f(x) = 0$, then $g(x) = 0$ but as $h(x), k(x)$ are both well-defined continuous function so $\lim_{y \to x} \frac{f(y)}{g(y)}$ exists and is not zero (because $\lim_{y \to x} \frac{g(y)}{f(y)}$ also exists), and since both limits exist and are finite quantity so their product is 1 and so at any $f(x) = 0$ we still have $h(x)k(x) = 1$.

4,11G Let M be finitely generated with basis $\{e_1, \ldots, e_n\}$. Then for each i, there exists $a_{i,j} \in I$ such that

$$\phi(e_i) = \sum_{j=1}^{n} a_{i,j} e_j$$

and so ϕ can be represented by a matrix with entries $a_{i,j}, i, j = 1, \ldots, n$ then it is basically Cayley-Hamilton theory. Let $A = \operatorname{adj}(tI_n - \phi)$ and $(tI_n - \phi)A = \det(A) I_n$. Write A as $A = \sum_{i=0}^{n-1} t^i A_i$ and $\det(A) = f(t)$ so we have

$$(tI_n - \phi) \sum_{i=0}^{n-1} t^i A_i = t^n A_{n-1} + \sum_{i=1}^{n-1} t^i (A_{i-1} - \phi A_i) - \phi A_0.$$

Let $f(t) = \sum_{i=0}^{n} t^i a_i$ and then compare coefficients for t^i we have $A_{n-1} = I_n$, $A_{i-1} - \phi A_i = a_i I_n, 1 \leq i \leq n-1$ and $-\phi A_0 = a_0 I_n$ and then for each $n - 1 \geq i \geq 0$ (tidy up the equations) we have

$$-\phi^{i+1} B_i = \phi^i a_i + \cdots + c_0$$

and take $i = n - 1$. Alternatively one can rewrite $\phi(e_i) = \sum_{j=1}^{n} a_{i,j} e_j$ as $(\phi \delta_{i,j} - a_{i,j}) e_j$ and let A be the matrix over the ring $R[\phi]$ with entries $A_{i,j} = \phi \delta_{i,j} - a_{i,j}$ and so $A \cdot e = 0$ which then implies the determinant of A is zero by multiplying the equation by $\operatorname{adj}(A)$ and this gives the required equation for ϕ.

Let ϕ be the identity map and $M = IM$ so $M \subset IM$. Let $a = 1 + a_{n-1} + \cdots + a_0$ so $a - 1 \in I$ and for all $m \in M$ we have

$$am = (1 + a_{n-1} + \cdots + a_0)m = (\phi^n + a_{n-1}\phi^{n-1} + \cdots + a_0)m = 0$$

so $aM = 0$.

Let $M = \mathbb{Z}/3\mathbb{Z}$ which is a $\mathbb{Z}$-module and take $I = \langle 2 \rangle$ so $IM = M$. Let $a = 3$ so $a - 1 \in I$ and $aM = 0$.

2013

3,1G If R is PID then for any ideal I, $I = \langle a \rangle$ for some $a \in I$ and so each element is given by $ka, k \in R$ and $\{a\}$ is a basis. As R is an integral domain the only zero divisor is 0 and so $ra = 0$ only if $r = 0$.

4,1G The first part is Orbit-Stabilizer theorem and the second part follows from the fact that G is a union of conjugacy classes and in this case conjugacy classes of size 1 corresponds to the elements in the center.

2,2G $\mathbb{Z}[i]$ is a Euclidean domain hence PID and so each ideal is generated by one element. Suppose $I_n = \langle a_n \rangle \subset I_{n+1} = \langle a_{n+1} \rangle$ then $a_{n+1} \big| a_n$ and for each element a_n there exists finitely many a_{n+1} which divides a_n because each element can be written as a product of irreducibles and so any ascending chain must terminate.

1,10G Let $\mathbb{R}^2$ be the usual vector space over $\mathbb{R}$ of dimension 2 and let X be the collection of 1-dimensional subspace and let

$$x_1 = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle, x_2 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle.$$

It is clear that if $g \in T$ is a multiple of scaler matrix then every element commutes with g so we take $g \in T$ which is not a multiple of scaler matrix and consider the elements in X which are fixed by g. Then if

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ dy \end{pmatrix} = k \begin{pmatrix} x \\ y \end{pmatrix}$$

for some $k \in \mathbb{R}$ then either $a = d$, which then implies g is a multiple of scaler matrix, or at least one of x, y is zero. Therefore the ones fixed by g are x_1, x_2. Let $h \in N$ then $hgh^{-1} \in T$ which is again not a multiple of scaler matrix, then $hgh^{-1}x_1 = x_1$ and hence $h^{-1}x_1$ is fixed by g. So $h^{-1}x_1 = x_1$ or x_2. Similarly, $hgh^{-1}x_2 = x_2$ implies $h^{-1}x_2 = x_1$ or $h^{-1}x_2 = x_2$. As h acts on X so $h^{-1}x_1 = x_1$, $h^{-1}x_2 = x_2$ or $h^{-1}x_1 = x_2$, $h^{-1}x_2 = x_1$. For the former case we conclude that $h \in T$ as the only elements which fix both x_1, x_2 are diagonal. For the other case we conclude that the entries of h satisfy the condition $a, d = 0$. Multiply h by a scaler if necessary we may assume h has the form

$$\begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}$$

and so every element in N lies in T or has the above form. By using

$$\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}$$

we conclude that N is generated by T and the desired matrix. The quotient just contains 2 elements.

For the second part we do exactly the same (we only used the fact every non-zero element is invertible in $\mathbb{R}$ and so the argument is exactly the same for finite field). So N is the subgroup which consists of elements of the form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$$

and so we have $2(p-1)^2$ of them.

4,11G Let $S = \{e_1, \ldots, e_k\}$ and relabel e_i if necessary we assume that $T = \{e_1, \ldots, e_l\}, l \leq k$. If $l = k$ then there is nothing to prove. If not, then for each i with $l < i \leq k$ there exists $a_{i,1}, \ldots, a_{i,l}, a_i \in R$ with $a_i \neq 0$ such that

$$a_{i,1}e_1 + \cdots + a_{i,l}e_l + a_i e_i = 0$$

by maximality of T. Thus $a_i e_i = -a_{i,1}e_1 - \cdots - a_{i,l}e_l \in N$. Now take $r = \prod_{i=l+1}^{k} a_i$ and so $re_i \in N$ for all $l < i \leq k$ and hence $rx \in N$ for all $x \in M$ and $r \neq 0$ because $a_i \neq 0$ and R is an integral domain.

For the second part take the map $f : M \to N$ by $f(x) = rx$. As r is non-zero by the first part and M is torsion free we conclude that the kernel is trivial. Therefore f is an injection so M is embedded into a submodule of N where N is a finitely generated free module.

3,11G As $\mathbb{C}[X, Y]$ is a UFD so if show $-X^3 + Y^2$ is an irreducible element, then the ideal generated by $-X^3 + Y^2$ is prime. Suppose it is not irreducible, and so we have $X^3 - Y^2 = fg$ for some non-unit elements f, g. Now we view f, g in $(\mathbb{C}[X])[Y]$ and suppose any of them is constant in Y, then $-X^3 + Y^2$ is divisible by some polynomial in $\mathbb{C}[X]$, which is impossible because for example, if it is divisible by $h \in \mathbb{C}[X]$, then $-X^3 + Y^2$ evaluated at $Y = 0$ or $Y = 1$ is again divisible by $h \in \mathbb{C}[X]$ but X^3 and $X^3 - 1$ are coprime. So we conclude that f, g must both have degree 1 in Y and we may assume they are both monic and so they have the form

$$f = Y + F(X), g = Y + G(X)$$

where we must have $F(X) + G(X) = 0$ and $F(X)G(X) = X^3$, and so we only have two cases to check, which are $F(X) = 1, G(X) = X^3, F(X) = X, G(X) = X^2$ but neither gives $F(X) + G(X) = 0$, and so $-X^3 + Y^2$ is irreducible.

I is not maximal, for example $J = \langle X, X^3 - Y^2 \rangle$ contains I and $J \neq R$ because $Y \notin J$ and $J \neq I$ because $X \notin I$ (by considering degree of X). Finally, let $J_t = \langle X + t, X^3 - Y^2$ for any $t \in \mathbb{C}$ then $J_t \neq R$ because J_t does not contain Y (again by considering the lowest degree, as Y has degree 1 and if we want to have some monomial involving Y in J_t we must multiply $X + t$ by some polynomial which is not constant in Y, and then this has degree bigger than 1). $J_t \neq I$ because $X + t \notin I$ (by considering the degree in X). Further for any $t \neq s$, $J_t \neq J_s$ because if so then $X + t - (X + s) = t - s \in J_t$ and $t - s$ is a unit which then implies $J_t = R$.

2,11G Let P_i be any polynomial of degree n, then $\mathbb{C}[X]/P_i$ is a $\mathbb{C}$-vector space of dimension n with basis $1, \ldots, X^{n-1}$. In this case it is not necessary to run through the Smith Normal form argument as we can see each P_i must be $X + 2$ by the relation $(X - 2)^4 x = 0$. Let $P = X - 2$ and so we have the following possibilities,

$$\mathbb{C}[X]/(P)^{\oplus 4}, \mathbb{C}[X]/(P)^{\oplus 2} \oplus \mathbb{C}[X]/(P^2), \mathbb{C}[X]/(P^2)^{\oplus 2}, \mathbb{C}[X]/(P^4).$$

Note that they are all distinct (for example, $A = \mathbb{C}[X]/(P)^{\oplus 2} \neq B = \mathbb{C}[X]/(P^2)$ as for each $a \in A$ we have $Pa = 0$ but this is not the case for $b \in B$).

For the last part as $(X+1)(X+i)(X-i) = X^3+1$ and every element in M can be represented by a quadratic polynomial $F(X) = aX^2 + bX + c$ so define a module homomorphism by

$$f : M \to \mathbb{C}[X]/\langle X \rangle \oplus \mathbb{C}[X]/\langle X + i \rangle \oplus \mathbb{C}[X]/\langle X - i \rangle, f(F) = (\bar{F}_X, \bar{F}_{X+i}, \bar{F}_{X-i})$$

where $\bar{F}_G$ means F mod G for any polynomial G. f is injective because if F is divisible by $X, X + i, X - i$ then it is divisible by $X^3 + 1$ as those three are coprime. f is surjective because

for any (x, y, z) in the image (where each element in the image can be represented by a complex number as they are quotient by degree 1 polynomials), we compute

$$1 \cdot X^2 + 1 \equiv 1 \bmod X, \frac{1}{2}X^2 - iX \equiv 1 \bmod X + i, -\frac{1}{2}X^2 + iX \equiv 1 \bmod X - i$$

and therefore take

$$F = x(X^2 + 1) + \frac{y}{2}(X^2 - iX) - \frac{z}{2}(X^2 + iX) = X^2(x + \frac{y}{2} - \frac{z}{2}) + X(-\frac{iy}{2} - \frac{iz}{2}) + x$$

and we have $f(F) = (x, y, z)$. So f is an isomorphism. Finally, we have a natural isomorphism

$$g : \mathbb{C}[X]/\langle X\rangle \oplus \mathbb{C}[X]/\langle X + i\rangle \oplus \mathbb{C}[X]/\langle X - i\rangle \to \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$$

as each element in the left hand side can be identified with a triple of complex numbers. Therefore f is the isomorphism with inverse given above.