1A Analysis Example Sheet 1

zc231

1. Let $a_{2k+1} = 1, a_{2k} = 0$ and $b_{2k+1} = 0$ and $b_{2k} = 1$.

2. For any real number l, we have, there exists $\epsilon = 1/2$ and for all $N > 0$, there exists $n > N$ such that $|a_n - l| > 1/2$ because if $l < 1/2$ then for any $N > 0$ we pick a_n with $a_n = 1$ (i.e. $n = 2^k$) and if $l \geq 1/2$ then we pick $n \neq 2^k$ so that $a_n = 0$ and $|a_n - l| > 1/2$. Since this is true for any real number l so a_n does not converge to any number.

3. (i) For all $-K$ ($K \geq 0$) we can find N such that $a_n \leq -K$ for all $n \geq N$

   (ii) This is trivial as $a_n \leq -K$ if and only if $-a_n \geq K$.

   (iii) For all $\epsilon > 0$ let $K = 1/\epsilon$ and so there exists N such that for all $n \geq N$, $a_n \geq K$, which is the same as $\frac{1}{a_n} \leq \frac{1}{K} = \epsilon$.

   (iv) No, e.g. $a_n = (-1)^n \frac{1}{n}$.

4. Induction on n. As $a_1 > b_1$ so $a_2 = (a_1 + b_1)/2 < a_1$, and

$$b_2 - b_1 = \frac{b_1(a_1 - b_1)}{a_1 + b_1} > 0, \frac{a_2}{b_2} = \frac{(a_1 + b_1)^2}{4a_1 b_1} > 1.$$

This proves the case $n = 1$. Suppose $a_n > a_{n+1} > b_{n+1} > b_n$, then it is clear that $a_{n+2} = \frac{a_{n+1} + b_{n+1}}{2} < a_{n+1}$ and

$$b_{n+2} - b_{n+1} = \frac{b_{n+1}(a_{n+1} - b_{n+1})}{a_{n+1} + b_{n+1}} > 0, \frac{a_{n+2}}{b_{n+2}} = \frac{(a_{n+1} + b_{n+1})^2}{4a_{n+1} b_{n+1}} > 1.$$

It is clear both limits exist as a_n is increasing and bounded above by b_1 and b_n is decreasing and bounded below by a_1. Let $c_n = a_n - b_n$ then we compute

$$c_{n+1} = \frac{(a_n - b_n)^2}{2(a_n + b_n)} = \frac{c_n^2}{2(c_n + 2b_n)} < \frac{c_n^2}{2c_n} = \frac{c_n}{2}.$$

Thus we conclude $c_n \to 0$ and so $(a_n)_n, (b_n)_n$ tend to the same limit. Note that for all n $a_n b_n = a_{n+1} b_{n+1}$ and thus for all n we have $a_n b_n = a_1 b_1$. This shows that the limit is $\sqrt{a_1 b_1}$.

5. Let $I_N = \cap_{n=1}^{N} [a_n, b_n]$. We conclude that I_N a is non-empty closed interval or a singleton for all N. It is clearly true for $N = 1$. Now suppose this is true for I_N. If I_N is a non-empty closed interval then let $I_N = [x, y]$. $I_{N+1} = [x, y] \cap [a_{N+1}, b_{N+1}]$. Suppose I_{N+1} is empty then this means either $a_{N+1} > y$ or $b_{N+1} < x$. By assumption $I_N = [x, y]$ so both x and y must be endpoints of some intervals, and then intersection of one of those intervals with $[a_{N+1}, b_{N+1}]$ must be empty which is a contradiction. Thus I_{N+1} is again a non-empty closed interval or a

singleton. If I_N is a singleton then again this singleton must be some endpoint of some interval so by the same argument I_{N+1} is non-empty. Thus I_N is a non-empty closed interval or a singleton for all N.

As $\mathbb{R}$ is compact so $\cap_{n=1}^{\infty}[a_n, b_n] =$ implies some finite intersection of intervals is empty and hence there exists N such that I_N is empty, which is a contradiction.

6. No, for example consider $(a_i, b_i) = (0, \frac{1}{n})$ and the intersection is empty. To see this, suppose x lies in the intersection, then clearly $x > 0$ and so $x > \frac{1}{n}$ for some n then $x \notin (a_n, b_n)$.

   One condition you can add is if we write $(a_n) \to a$, $(b_n) \to b$ then we require $a < b$. Clearly $(a_n), (b_n)$ both converge as they are monotonic and bounded so any points $x \in (a, b)$ will be in the intersection.

7. By B-W, we have a convergent subsequence say $a_{i_1}, a_{i_2}, \ldots$ which converges to a. As the sequence does not converge, so there exists $\epsilon > 0$ such that for each N there exists $K(N)$ (constant depending on N) such that $|a_{K(N)} - a| > \epsilon$. The sequence $(a_{K(N)})_N$ is clearly bounded and so we apply B-W to get a convergent subsequence with limit b. Clearly $|b-a| > \epsilon$ so $b \neq a$.

   For the second part, suppose (a_n) has no convergent subsequence, then by B-W (a_n) must be unbounded and thus by definition for all $k > 0$, there exists n_k with $|a_{n_k}| > k$ and thus $|a_{n_k}| \to \infty$. This then means a_{n_k} has a subsequence which converges to ∞ or $-\infty$.

8. Suppose not then there exists $\epsilon > 0$ such that for all $N > 0$ there exists $n_N > N$ with $|a_{n_N} - a| > \epsilon$. Let N run through natural numbers and thus (a_{n_N}) is a subsequence but clearly a_{n_N} has no subsequence convergent to a because $|a_{n_N} - a| > \epsilon$.

9. For each $k \geq 1$ there exists N_k such that for all $m, n > N_i$, $|a_m - a_n| < 2^{-k}$. Then clearly $N_j \geq N_j$ for $j > i$ (as $2^{-j} < 2^{-i}$). Then we pick $a_{n_k} = a_{N_k+1}$.

10. As $f(x) \in (0, \infty)$ so $a_{n+1} = a_n + f(a_n) > a_n$. We prove by induction that $a_{n+1} \geq 1 + nf(a_n)$. For $n = 1$ we have $a_2 = 1 + f(a_1) \geq 1 + f(a_2)$ because $a_1 < a_2$. Suppose $a_n \geq 1 + (n-1)f(a_{n-1})$ then

$$a_{n+1} = a_n + f(a_n) \geq 1 + (n-1)f(a_{n-1}) + f(a_n) \geq 1 + (n-1)f(a_n) + f(a_n) = 1 + nf(a_n)$$

   as a_n is increasing.

   Now suppose a_n is bounded above by k and let $c = f(k)$ which is a fixed positive number. Pick n large enough so that $1 + nc > k$ which is possible as k and c are both fixed. Therefore

$$a_{n+1} \geq 1 + nf(a_n) \geq 1 + nf(k) = 1 + nc > k$$

   which is a contradiction. Thus, a_n is bounded and so $a_n \to \infty$.

11. Since $|\sin n| \leq 1$ and so $\sum \frac{\sin n}{n^2}$ converges absolutely. For $\sum \frac{n^2 z^n}{5^n}$ we need $|z| < 5$ by ratio test. The third one converges by alternating test. For the fourth one we get $|z| < 1$ by ratio test and also on the boundary if $z = 1$ the sum also converges (to 0). The last one converges by ratio test (the ratio is e^{-1}).

    For the second one, on $|z| = 5$ the series diverges because each term is unbounded on $|z| = 5$. For the fourth one, on $|z| = 1$, if $z = 1$ then the series converges. We show that in fact it converges for all z with $|z| = 1$. Let $z \neq 1$, $|z| = 1$ and so $(z-1)$ is a non-zero quantity so it suffices to show $\sum_n \frac{e^{in\theta}}{n}$ converges for any θ with $e^{in\theta} \neq 1$. For any m we have

$$|\sum_{n=1}^{m} e^{in\theta}| = \left|\frac{e^{im\theta}-1}{e^{i\theta}-1}\right| \leq \frac{2}{|e^{i\theta}-1|}$$

and so it is bounded by some constant C. Let $s_m = \sum_{n=1}^{m} e^{in\theta}$ then we have $\frac{e^{im\theta}}{m} = \frac{s_m - s_{m-1}}{m}$ and so for any $k > n$,

$$\begin{aligned}
\left|\sum_{m=n}^{k} \frac{a_m}{m}\right| &= \left|\sum_{m=n}^{k} \frac{s_m}{m} - \sum_{m=n}^{k} \frac{s_{m-1}}{m}\right| \\
&= \left|\sum_{m=n}^{k} \frac{s_m}{m} - \sum_{m=n}^{k-1} \frac{s_m}{m+1} - \frac{s_{n-1}}{n}\right| \\
&\leq \left|\frac{s_k}{k}\right| + \left|\frac{s_{n-1}}{n}\right| + \sum_{m=n}^{k-1}\left|\frac{s_m}{m(m+1)}\right| \\
&\leq \frac{2C}{n} + \sum_{m=n}^{k-1} C\frac{1}{m(m+1)} \\
&\leq \frac{2C}{n} + \frac{C}{n} - \frac{C}{k} \leq \frac{3C}{n}.
\end{aligned}$$

This is true for all $k < n$ and so the series is Cauchy (let n be big enough) and hence the series converges (as $\mathbb{C}$ is complete).

12. $s_{2n} = 1 - \frac{1}{2} + \ldots + \frac{1}{2n-1} - \frac{1}{2n}$ and

$$H_{2n} - H_n = 1 + (\frac{1}{2} - 1) + \ldots + \frac{1}{2n-1} + (\frac{1}{2n} - \frac{1}{n}) = s_{2n}$$

because we are free to change the order of finite sum. $t_{3n} = 1 + \frac{1}{3} - \frac{1}{2} + \ldots + \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n}$ and

$$H_{4n} - \frac{1}{2}H_{2n} = (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} - \frac{1}{4}) + \ldots + (\frac{1}{4n-3} + \frac{1}{4n-2} + \frac{1}{4n-1} + \frac{1}{4n} - \frac{1}{4n-2} - \frac{1}{4n})$$

which is $t_{3n} + \frac{1}{2}H_n$.

$(s_n)_n$ converges by alternating test, say s. t_{3n} also converges by alternating test and the fact

$$|-\frac{1}{2n-2}| > \frac{1}{4n-3} + \frac{1}{4n-1} > |-\frac{1}{2n}|.$$

Since $|t_{3n+1} - t_{3n}| = \frac{1}{4n+1} < \frac{1}{4n}$ and $|t_{3n+2} - t_{3n}| = \frac{1}{4n+3} < \frac{1}{4n}$ so by using triangle inequality we conclude that $(t_n)_n$ converges. Finally, as $t_{3n} = s_{4n} + \frac{1}{2}s_{2n}$ we conclude t_{3n} has limit $s + \frac{1}{2}s = \frac{3}{2}s$ which is the same as the limit of t_n.

13. For all $\epsilon > 0$, there exists N_0 such that for all $n > N_0$, $|a_n - a| < \epsilon/2$. Let k be the maximum of $|a_1 - a|, \ldots, |a_{N_0} - a|$, and pick $N > 2N_0k/\epsilon - N_0$ (basically N large enough), then we have for all $n > N$,

$$|b_n - a| = |\frac{a_1 - a}{n} + \ldots + \frac{a_n - a}{n}| \leq |\frac{a_1 - a}{n}| + \ldots + |\frac{a_n - a}{n}| = \sum_{k=1}^{N_0} |\frac{a_k - a}{n}| + \sum_{k > N_0} |\frac{a_k - a}{n}|.$$

The first part of the sum is at most kN_0/n and the second part is at most $(n - N_0)\epsilon/2n$, and so we conclude $|b_n - a| < \epsilon$ by the condition on n.

14. By Cauchy condensation test, $\sum \frac{1}{n \log n}$ converges if and only if $\sum \frac{2^n}{2^n (\log 2^n)^\alpha}$ converges. As $(\log 2^n)^\alpha = n^\alpha (\log 2)^\alpha$ and $(\log 2)^\alpha$ is just a constant so we consider the sum $\sum \frac{1}{n^\alpha}$ and clearly this converges if and only if $\alpha > 1$.

    Apply condensation test we consider the sum $\sum \frac{1}{n \log 2 \log \log 2^n}$. As $\log 2$ is a constant and $\log \log 2^n = \log n + \log \log 2$ we consider the sum $\sum \frac{1}{n(\log n + \log \log 2)}$ then we can again use condensation test so we consider the sum $\sum \frac{1}{n \log 2 + \log \log 2}$ and this is clearly divergent.

15. Let s_n be the partial sum of the first n terms and let $b_n = a_n / s_n$. It is clear that $b_n / a_n \to 0$ because $s_n \to \infty$ (as a_n all positive so the sum diverges to infinity). Let t_n be the partial sum of b_n.

    Now consider for $n < m$,

$$t_m - t_n = \frac{a_m}{s_m} + \ldots + \frac{a_{n+1}}{s_{n+1}} > \frac{a_m}{s_m} + \ldots + \frac{a_{n+1}}{s_m} = \frac{s_m - s_n}{s_m} = 1 - \frac{s_n}{s_m}.$$

    Fix n and we see as $m \to \infty$, $s_m \to \infty$ so $t_m - t_n \to 1$ (for n fixed). Thus, for each n there exists $N(n)$ (constant depending on n) such that $t_{N(n)} - t_n > \frac{1}{2}$ and hence t_n diverges.

16. Note that as $\sum a_n$ converges but not absolutely converges so the sequence a_n cannot be all positive or negative and further a_n has infinitely many positive terms and infinitely many negative terms. Further, assume $a_n \neq 0$ for all n because 0 does not do any contribution to the sum.

    Define the sequence $b_n = a_n$ if $a_n > 0$ and $b_n = 0$ otherwise, $c_n = -a_n$ if $a_n < 0$ and $c_n = 0$ otherwise. Note that $\sum b_n = \sum c_n = \infty$ (i.e. the sum converges to infinity) because

$$\infty = \sum_n |a_n| = \sum_n b_n + \sum_n c_n, \sum_n a_n = \sum_n b_n - \sum_n c_n$$

    where the first equality implies at least one of $\sum b_n$,$\sum c_n$ must tend to infinity and the second equality shows that if only one of them tend to infinity then $\sum_n a_n$ does not converge, which is a contradiction.

    Without loss of generality, let $x > 0$. Take the smallest n_1 with $b_1 + b_2 + \cdots + b_{n_1} > x$. Then take the smallest n_2 with $b_1 + \cdots + b_{n_1} - c_1 - \cdots - c_{n_2} < x$. Repeat the above process and we reorder the sum in this way (and we can because $\sum b_n, \sum c_n$ both tend to infinity so n_i is well-defined for all i), say $\sum d_n$, which looks like

$$b_1 + \cdots + b_{n_1} - c_1 - \cdots - c_{n_2} + b_{n_1+1} + \cdots + b_{n_1+n_3} - c_{n_2+1} + \cdots - c_{n_2+n_4} + \cdots$$

    Now we show that $\sum d_n$ indeed tends to x. Let N be an integer with $n_1 + \cdots + n_{k-1} \leq N < n_1 + \cdots n_k$, then the difference between x and $\sum_{n=1}^{N} d_n$ is bounded by b_{N_1} if k is even and c_{N_2} if k is odd where $N_1 = n_1 + n_3 + \cdots + n_{k-1}$ and $N_2 = n_2 + n_4 + \cdots + n_{k-1}$. Finally as $\sum a_n$ converges so $a_n \to 0$ and hence $b_n, c_n \to 0$ as $n \to \infty$. Therefore, for each $\epsilon > 0$ we just need to pick N large enough with $b_{N_1}, c_{N_2} < \epsilon$ so that the difference between x and $\sum_{n=1}^{N}$ is bounded by ϵ.

17. It is divergent. One can prove this by integration test. So we compute the integration

$$\int_1^\infty \frac{1}{f(x)} dx = \sum_{k=0}^{\infty} \int_{e_k(1)}^{e_{k+1}(1)} \frac{1}{f(x)} dx$$

where $e_k = \exp(\exp(\cdots))$ which is the k-th iteration of exponential function and $e_0 = 1$. Then between $e_k(1)$ and $e_{k+1}(1)$ we have

$$\frac{1}{f(x)} = \frac{1}{x\log(x)\cdots\log_k(x)}.$$

Let $y = \log_k(x)$ then $x = e_k(y)$ and $dx = e_k(y)e_{k-1}(y)\cdots e_1(y)$. Therefore,

$$\int_{e_k(1)}^{e_{k+1}(1)} \frac{1}{f(x)}dx = \int_1^e \frac{dy}{y} = 1.$$

Summing over $k = 1$ to ∞ we see it diverges.

18. No. Suppose $(0,1) = \cup_{i\in S} I_i$ where S is a set (note S is not necessarily countable, although in this case it must be). Take any $A_1, B_1 \in \{I_i : i \in S\}$ and without loss of generality we assume A_1 is on the left of B_1. We construct two sequences of intervals by the following. Given $A_n = [a_n, b_n], B_n = [c_n, d_n]$, let A_{n+1} be the unique interval which contains $\frac{b_n+c_n}{2}$ (it is unique because the intervals are disjoint) and let B_{n+1} be any interval which lies between A_{n+1} and B_n (which exists as A_{n+1} is disjoint from B_n). Now the sequence (b_n) is increasing by construction and is bounded above by c_1 so $b_n \to b$ for some b. Similarly (c_n) is decreasing and bounded below by b_1 so $c_n \to c$ for some c. Finally, if $d_n = c_n - b_n$, then

$$d_{n+1} = c_{n+1} - b_{n+1} < c_n - b_{n+1} < c_n - \frac{b_n + c_n}{2} = \frac{c_n - b_n}{2} = \frac{d_n}{2}$$

because by construction $b_{n+1} \geq \frac{b_n+c_n}{2}$, and so $d_{n+1} < \frac{1}{2^n}d_1$ where d_1 is a constant and $d_n > 0$ for all n so $d_n \to 0$ which then implies $b = c$.

Suppose b is contained in any interval I, then there exists $\epsilon > 0$ such that either I contains $[b-\epsilon, b]$ or $[b, b+\epsilon]$ as I is an interval and we assume I contains $[b-\epsilon, b]$ then as $b_n \to b$ there exists N large enough so that $|b_N - b| < \epsilon$ which then implies that $A_N \cap I$ is non-trivial. For the case $[b, b+\epsilon]$ as $c = b$ there exists c_N with $|c_N - b| < \epsilon$ so $B_N \cap I$ is non-trivial.

19. Let S_n be the partial sum. Then we can prove by induction (which is just a direct computation) on n to check

$$s_n = \frac{\sum_{k=1}^{2^n-1} z^k}{1 - z^{2^n}} = \frac{z(1 - z^{2^n-1})}{(1 - z^{2^n})(1 - z)}.$$

Then

$$|s_n - \frac{z}{1-z}| = \left|\frac{z}{1-z}\right| |z^{2^n-1}| \left|\frac{z-1}{1-z^{2^n}}\right|,$$

and as $|z| < 1$ so there exists n such that $|1 - z^{2n}| > \frac{1}{2}$ by triangle inequality and so the term $|\frac{z-1}{1-z^{2^n}}|$ is bounded by a constant and as $|z^{2^n-1}| \to 0$ for n large enough so we conclude that $s_n \to \frac{z}{1-z}$.

If $|z| > 1$ then we have

$$\left|s_n - \frac{1}{1-z}\right| = \left|\frac{1}{1-z^{2^n}}\right|$$

and this tends to 0 as $|z| > 1$. So $s_n \to \frac{1}{1-z}$.

For $|z| = 1$ it does not converge because if we let $z = e^{2\pi i t}$ and substitute this into the general form of s_n then the sum does not converge (another reason is if it converges then as the sum is continuous as a function in z so if it converges at some $|z| = 1$ then $\frac{z}{z-1}$ must be equal to $\frac{1}{z-1}$ at that point but this will give $z = 1$).