1A Analysis Example Sheet 2

zc231

IVT stands for intermediate value theorem and MVT stands for mean value theorem.

1* f is continuous at $\frac{1}{2}$. If h is rational, $f(\frac{1}{2}+h)-f(\frac{1}{2})=h$ and if h is irrational then $f(\frac{1}{2}+h)-f(\frac{1}{2})=-h$ and thus for all $\epsilon>0$, pick $\delta=\epsilon$ so that for all $|h|<\delta$, $|f(\frac{1}{2}+h)-f(\frac{1}{2})|=h<\epsilon$ and so it is continuous at $\frac{1}{2}$.

f is not continuous at any other point. Let $a\neq\frac{1}{2}$. Suppose a is rational. Let $|1-2a|=r$, then there exists $\epsilon=\frac{r}{2}$, such that for all $\delta>0$, pick $|h|$ small enough (say $|h|<r/2$) and h irrational, then $|f(a+h)-f(a)|=|1-2a-h|>\frac{r}{2}=\epsilon$. Similarly if a is irrational, we repeat the above but pick h with $a+h$ rational in the last step.

1. Suppose $a_n\to a$ and if $a_n=a$ for all a then clearly $b_n=a$ for all a and so $a_n=b_n$. Suppose a_n is not constant. If $(a_n),(b_n)$ are different pick the least n_1 such that $a_{n_1}\neq b_{n_1}$ then by assumption we have $n_1<n_2\leq n_3<n_4\cdots$ with

$$a_{n_1}=b_{n_2}=a_{n_3}=b_{n_4}=\cdots$$

and the terms $a_{n_1},a_{n_3},\ldots$ and so $a_{n_i}-a$ is fixed so a_n cannot tend to a which is a contradiction.

2* Suppose $f(x)\to\infty$ as $x\to\infty$. Given any sequence $x_n\to\infty$, we have for all $K>0$, there exists M such that $f(x)>K$ for all $x>M$ and for each such M, there exists N such that $x_n>M$ for all $n>N$ and thus $f(x_n)>K$.

Conversely, suppose $f(x)\not\to\infty$ as $x\to\infty$, this is saying there exists $K>0$ such that for all $M>0$ there exists $x>M$ such that $f(x)<K$. Set $M=1,2,\ldots$ and so we have a sequence $(x_m)_m$ with $x_m>m$ but $f(x_m)<K$. Clearly the condition $x_m>m$ implies that $x_m\to\infty$ but $f(x_m)<K$ so $f(x_m)\not\to\infty$ so we have a sequence $x_m\to\infty$ but $f(x_m)\not\to\infty$.

2. H is not continuous at 0. Take $\epsilon=\frac{1}{4}$ and so for all $\delta>0$, there exists h such that $|H(h)-H(0)|>\frac{1}{4}$ (either $H(0)>\frac{1}{2}$ or $H(0)\leq\frac{1}{2}$ and this holds for both cases). Similarly $H(0)\neq 1$ or $H(0)\neq 0$. Assume $H(0)\neq 0$ then take a sequence $x_n\to 0$ from the left and then $H(x_n)=0$ for all n but $0\not\to H(0)$.

3. Let $f(x)=|x|$ and so $f(x)\to 0$ as $x\to 0$. Let $g(x)=1$ if $x\geq 0$ and 0 if $x<0$, then $g(x)\to 0$ as $x\to 0$ (from the left).Then $g(f(x))=g(|x|)=1$ for all x (as $|x|\geq 0$) and so $g(f(x))\not\to 0$ as $x\to 0$.

4. For each n, $h_{n+1}(x)=\max\{h_n(x),f_{n+1}(x)\}$. $h_1(x)$ is continuous. Suppose $h_n(x)$ is continuous, then

$$h_{n+1}(x)=\max\{h_n(x),f_{n+1}(x)\}=\frac{h_n(x)+f_{n+1}(x)}{2}+\frac{|h_n(x)-f_{n+1}(x)|}{2}.$$

As the sum and difference of continuous functions are again continuous and if $f(x)$ is continuous, so is $|f(x)|$ then we conclude that $h_{n+1}(x)$ is continuous. Thus by induction $h_n(x)$ is continuous for all x.

No. Consider $f_n(x) = 2^n x$ for $x \leq \frac{1}{2^n}$ and $f_n(x) = 1$ for $x > \frac{1}{2^n}$. Then it is clear that each f_n is continuous. Now $h(0) = 0$ and $h(x) = 1$ for $x > 0$ which is not continuous.

5. If $g(0) = 0$ or $g(1) = 1$ then we are done, otherwise let $f(x) = g(x) - x$ which is continuous. By assumption $g(0) > 0$ and $g(1) < 1$ so $f(0) > 0$ and $f(1) < 0$ so by IVT there exists c such that $f(c) = 0$ which is the same as $g(c) = c$.

   Let $h(0) = 1$ and $h(1) = 0$ and $h(x) = x^2$ for $0 < x < 1$ then clearly it is bijective on $[0, 1]$ but $h(x)$ has no fixed point as $x^2 = x$ implies $x = 0$ or 1.

   Let $p(x) = x^2$ for $x \in (0, 1)$ then clearly this is continuous but $p(x) \neq x$ for all $x \in (0, 1)$.

6* Define g on $[0, \theta]$ by $g(\theta) = f(\theta) - f(\pi+\theta)$ then g is continuous. Then the points $e^{i\theta}$ and $e^{i(\pi+\theta)}$ are diametrically opposite. $g(0) = f(0) - f(\pi)$ and $g(\pi) = f(\pi) - f(2\pi)$. Since $f(0) = f(2\pi)$ we have $g(\pi) = -g(0)$. Thus, either $g(0) = g(\pi) = 0$ in which case we have $f(0) = f(\pi)$ or $g(\pi), g(0)$ have opposite sign in which case we apply IVT so that there exists c such that $g(c) = 0$ and so $f(c) = f(\pi + c)$.

6. Let $f(x) = 2x^5 + 3x^4 + 2x + 16$ then $f'(x) = 2(5x^4 + 6x^3 + 1) = 2(x + 1)(5x^3 + x^2 - x + 1)$. Let $g(x) = 5x^3 + x^2 - x + 1$. We want to determine the local maximum and minimum of f so we consider the point at which g vanishes. $g'(x) = 2(3x + 1)(5x - 1)$ and by considering $g''(x)$ we see g has a local maximum at $-1/3$ and local minimum at $1/5$ and $g(-1/3), g(1/5)$ are both positive which means $g(x)$ only has one real root say t. Further, $g(-1) = -2 < 0$ and $g(0) > 0$ so $t \in (-1, 0)$ and therefore $f'(x)$ has two real roots. Then by considering $f''(x)$ or the fact $f(x) \to -\infty$ as $x \to -\infty$ we see that -1 is a local maximum and t is a local minimum. Finally, as $t > -1$, so if $-1 < t < 0$ then

   $$f(t) = 2t^5 + 3t^4 + 2t + 16 > -2 + 0 - 2 + 16 > 0$$

   and if $t > 0$ then clearly $f(t) > 0$ so we conclude $f(x) > 0$ for all $x > -1$. Now since $f(-1) > 0, f(-2) < 0$ so there is a real root in $[-2, -1]$ and it is the only one because if there is another one then one would have another local maximum or minimum. To see this, suppose $f(z_1) = f(z_2) = 0$ then by MVT there is a point between z_1 and z_2 at which the derivative of f vanishes.

7. As $x = 0, y = 0$ is a solution and if (x, y) is a solution, so is $(-x, -y)$ so we consider the solutions with $x > 0$. It is the same as finding x with $x = \sin(6\pi \sin(6\pi x))$ because for each x satisfying the above relation we take $y = \sin(6\pi x)$ and conversely if $y = \sin(6\pi x), x = \sin(6\pi y)$ then $x = \sin(6\pi \sin(6\pi x))$. Write $g(x) = \sin(6\pi \sin(6\pi x))$ and $f(x) = g(x) - x$.

   For each $0 < x < \frac{1}{12}$, $\sin(6\pi x)$ is monotonic and we obtain 3 cycles in this range for $\sin(6\pi \sin(6\pi x))$. Consider take t_1 to be the first point such that $g(t_1) = 1$. We claim that there is no solution in $(0, t_1)$. We know $f(0) = 0$ and by direct computation

   $$f'(x) = 36 \cos(6\pi \sin(6\pi x)) \cos(6\pi x) - 1$$

   which is monotonically decreasing on $[0, t_1]$ and therefore $f'(x)$ has at most one zero and as $f'(0) > 0$ and $f'(t_1) = -1 < 0$ so f' has a zero and as f' is decreasing $f'' \leq 0$ (you might need question 10) and so the point at which $f' = 0$ is a local maximum and hence the minimum of f is either $f(0)$ or $f(t_1)$ and as $f(t_1) > 0$ so we conclude that f has no zero in $(0, t_1)$.

   Then we consider t_2 which is the smallest number bigger than 0 such that $g(t_2) = 0$. Then clearly by IVT on (t_1, t_2) we have a solution for $f = 0$. Similarly then consider t_3 which is the next point at which $g = 0$ and t_4 which is the next point at which $g = 1$ then there is a solution for $f = 0$. Repeat this argument and we have 5 solutions in $(0, \frac{1}{12})$. Then repeat this argument for the intervals $(\frac{1}{12}, \frac{2}{12}), \ldots, (\frac{11}{12}, 1)$ we have 6 solutions in each of these intervals and hence we have 71 solutions in total for $x > 0$ and so we have 142 solutions in total for $x \neq 0$.

8* Suppose f is not continuous at some $t \in (a, b)$ and we prove f is bounded (hence a contradiction). There exists $\epsilon > 0$ such that for all $\delta > 0$ there exists h with $|h| < \delta$ such that

$$|f(t + h) - f(t)| > \epsilon.$$

Without loss of generality we assume $f(t+h)-f(t) > \epsilon$ (the other case when $f(t+h)-f(t) < -\epsilon$ is similar). Then let $x = t + 2h, y = t$ so that $(x + y)/2 = t + h$, by assumption we have

$$\epsilon < f(t + h) - f(t) < f(t + 2h) - f(t + h)$$

and hence $f(t + 2h) > f(t + h) + \epsilon > f(t) + 2\epsilon$. Then set $x = t + 3h, y = t + h$ etc. we prove inductively that $f(t + nh) > f(t + (n - 1)h) + \epsilon$ whenever $t + nh \in (a, b)$. To see this, suppose this is true for n then set $x = t + (n + 1)h, y = t + (n - 1)h$ then

$$\epsilon < f(t + nh) - f(t + (n - 1)h < f(t + (n + 1)h) - f(t + nh).$$

Therefore, we have $f(t + nh) > f(t) + n\epsilon$ for all n with $t + nh \in (a, b)$.

For all $M > 0$, pick n so that $f(t) + n\epsilon > M$ then we just need to pick δ carefully so that $t + n\delta \in (a, b)$ and so $t + nh \in (a, b)$ (note n, ϵ are independent of the choice of δ), and this gives $f(t + nh) > M$, which shows f is unbounded.

f does not need to be continuous at the endpoints. For example, consider f defined on $[0, 1]$ by $f(0) = 0$ and $f(x) = 1$ for $x \in (0, 1]$ then it is bounded and clearly $f((x + y)/2) = f(x)/2 + f(y)/2$ for all $(x, y) \in (0, 1)$ but f is not continuous at 0.

8. Since f is continuous on $[0, 1]$ so it attains its maximum and minimum, say m and n. Suppose $m \neq n$ (hence not both zero), and without loss of generality, say $m \neq 0$. Then consider the set

$$\{x \in [0, 1] : f(x) = m\}.$$

This is a non-empty set of real and hence it has a least upper bound, K. By definition we have a sequence $x_n \to K$, with $f(x_n) = m$ and since f is continuous, we have $f(x_n) \to f(K)$ so $f(K) = m$. Now if $K \neq 1$ then there exists δ such that $f(K) = (f(K - \delta) + f(K + \delta))/2$ but clearly, $f(K + \delta) < m$ and $f(K - \delta) \leq m$ which gives a contradiction and so $K = 1$. Finally, as $f(K) = m = f(1) = 0$ so we must have $m = 0$ which contradicts our assumption that $m \neq n$. Therefore, $m = n$ and f is constant so $f \equiv 0$.

9. Let $x = \frac{p}{q} \in \mathbb{Q}$ then take $\epsilon = \frac{1}{2q}$ so for any $\delta > 0$, pick h small enough that $x + h$ is irrational (which is always possible) then we have $|f(x + h) - f(x)| = \frac{1}{q} > \epsilon$. Now let $x \notin \mathbb{Q}$. Then for all $\epsilon > 0$, there exists n with $\frac{1}{n} < \epsilon$.

We consider the set of rational numbers around x with denominator less than n. If $\frac{p}{q} \in (x - \delta, x + \delta)$ then we have

$$qx - q\delta < p < qx + q\delta$$

and so in particular if $\delta < \frac{1}{2q}$ then the length of the interval $(qx - q\delta, qx + q\delta)$ is less than 1 and so we cannot pick any p in this interval and also for any $q' < q$ the length of the interval is smaller so we conclude that there exists no rational number with denominator less than or equal to q in $(x-\delta, x+\delta)$. So now pick $\delta < \frac{1}{2n}$ so each rational number $\frac{p}{q}$ in the interval has denominator bigger than n so $f(\frac{p}{q}) < \frac{1}{n} < \epsilon$. So for all $|h| < \delta$ we have $|f(x+h) - f(x)| < \epsilon$ and so f is continuous at x.

10. (i) True. f is differentiable everywhere so we can compute $f'(x)$ by using the right limit, i.e. $f'(x) = \lim_{h>0, h\to 0}(f(x+h) - f(x))/h$. Since f is increasing so $f(x+h) - f(x) \geq 0$ and hence $f'(x) \geq 0$.

    (ii) True. Suppose f is not increasing then there exists $x \in (a,b)$ and $h > 0$ (with $x+h \in (a,b)$ such that $f(x+h) - f(x) < 0$. By MVT, $f(x+h) - f(x) = hf'(z)$ for some $z \in (x, x+h)$ and since $f'(z) \geq 0$ so $f(x+h) - f(x) \geq 0$ which is a contradiction.

    (iii) False. Consider $f(x) = x^4$ on $[0,1]$.

    (iv) True, by using the same argument as in (ii).

11* (i) $f''(t)$ is the derivative of $f'(t)$ and so by previous question we know $f'(t)$ is increasing. Since $f'(0) > 0$ so $f'(t) > 0$ for all t and hence f is strictly increasing. Therefore, $f(t) > 0$ for all $t > 0$.

    Now if $f'(0) \geq 0$ then again by $f''(t) \geq 0$ we conclude that $f'(t) \geq 0$ for all $t > 0$ therefore, $f(t)$ is increasing and hence $f(t) \geq 0$ for all t. If $f(x) > 0$ at some $x \in (0,1)$, then by MVT there exists $z \in (x, 1)$ such that $f(x) - f(1) = (x-1)f'(z)$ and so $f'(z) < 0$ which is a contradiction. Hence $f(t) = 0$ for all $t \in [0,1]$.

    Finally if $f'(1) \leq 0$. Let $g(t) = f(1-t)$ then $g'(t) = -f'(1-t)$ and so $g'(0) = -f'(1) \geq 0$. Further, $g''(t) = f''(1-t) \geq 0$ and $g(0) = f(1) = f(0) = g(1)$. Thus by previous part we conclude g is constant and so is f.

    (ii) Suppose $f(x) > 0$ for some $x \in [0,1]$. Then by MVT, there exist $p, q \in [0,1]$ with $p \in (0,x)$ and $q \in (x, 1)$ such that

$$f(x) - f(0) = xf'(p), f(1) - f(x) = (1-x)f'(q).$$

    Since $f(0) = f(1) = 0$ so we conclude that $f'(p) > 0$ and $f'(q) < 0$. Now apply MVT on the function $f'(t)$ so there exists $r \in (p,q)$ such that

$$f'(q) - f'(p) = (q-p)f''(r).$$

    As $f'(p) > 0, f'(q) < 0, q > p$ we conclude that $f''(r) < 0$ which is a contradiction.

    (a) A direct computation shows $f(0) = f(1) = 0$. Also,

$$f''(t) = (b-a)^2 g''((1-t)a + tb) \geq 0$$

    as $g''(t) \geq 0$ for all $t \in [a,b]$. Therefore, by (ii) $f(t) \leq 0$ for all $t \in [0,1]$ which yields the desired inequality.

11. (i) By definition

$$g''(0) = \lim_{h\to 0} \frac{g'(h) - g'(0)}{h} = \lim_{h\to 0} \frac{g'(h)}{h} > 0$$

So there exists a small interval $(0, \delta)$, $\delta > 0$, such that for all $0 < h < \delta$, $g'(h) > 0$ because if not then we have a sequence $h_n \to 0$ from the right with $g'(h_n) \le 0$ then the limit will be non-positive and then take $x \in (0, \delta)$ then we have, by MVT, that

$$g(x) = g(x) - g(0) = xg'(z) > 0, z \in (0, x) \subset (0, \delta).$$

(ii) Let $g(x) = f(2x) - 2f(x)$ then $g(0) = 0$ and $g'(0) = 2f'(0) - 2f'(0) = 0$ and

$$g''(0) = 4f''(0) - 2f''(0) = 2f''(0) > 0$$

so we apply (i) and conclude that $g(x) > 0$ for some x, so $f(2x) > 2f(x)$.

12. Let $g(x) = f(x) - lx$ then $g(x)$ is differentiable for all x and $g'(x) \to 0$ as $x \to \infty$. To show $\frac{f(x)}{x} \to l$, it suffices to show $\frac{g(x)}{x} \to 0$. For all $\epsilon > 0$, there exist Y such that for all $x > Y$, $|g'(x)| < \frac{\epsilon}{2}$. Now for each $N > Y$, using MVT we have

$$g(N) - g(Y) = (N - Y)g'(z), \text{ for some } z \in (Y, N).$$

Then $|g(N) - g(Y)| < \frac{(N-Y)\epsilon}{2}$ and by triangle inequality we have

$$|g(N)| < |g(Y)| + \frac{(N - Y)\epsilon}{2}.$$

Therefore,

$$\left|\frac{g(N)}{N}\right| < \left|\frac{g(Y)}{N}\right| + \frac{(N - Y)\epsilon}{2N}.$$

As Y is fixed we write $|g(Y)| = M$, and so if we pick $N > \frac{2M}{\epsilon} - Y$ then the right hand side of the above is less than ϵ. This proves $\frac{g(x)}{x} \to 0$.

The converse is not true. For example, let $f(x) = \sin x$ then $f(x)/x \to 0$ as $x \to \infty$ but $f'(x) = \cos x$ does not tend to a limit.

13. Let $g : \mathbb{R} \to \mathbb{R}/\mathbb{Q}$ be the quotient map so basically this is the natural quotient map of abelian groups, so $g(x) = 0$ if x is rational and $g(x) = x \bmod \mathbb{Q}$ if x is irrational which means $g(x) = g(y)$ if and only if $x - y \in \mathbb{Q}$. Then as $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable so the quotient must again be uncountable and is bijective to $\mathbb{R}$ so let $h : \mathbb{R}/\mathbb{Q} \to \mathbb{R}$ be a bijection. Let $f = hg$. Then for all $a < b$ and $t \in \mathbb{R}$, take y with $h(y) = t$ and we want some $x \in (a, b)$ with $g(x) = y$. As $g(x) = g(x - q)$ for any rational q then we just take a suitable q such that $x - q \in (a, b)$.

Alternatively, we consider the density function. As $\mathbb{R}$ is bijective to $(0, 1)$ so it suffices to construct a function define on $(0, 1)$ such that for each $(a, b) \subset (0, 1)$ and $t \in (0, 1)$ we have $x \in (a, b)$ with $f(x) = t$. For each $x \in (0, 1)$ we take the decimal expansion of $x = 0.a_1a_2\ldots$ and let S_n be the number of $a_i, i = 1, 2, \ldots, n$ such that $a_i = 1$ and define $b_n = \frac{S_n}{n}$. Define $f(x) = \lim_{n\to\infty} b_n$ if the limit exists and if not we define $f(x) = \frac{1}{2}$. Then clearly f satisfies the condition because each real number is a limit of a rational sequence for example, if $\alpha = \frac{1}{\sqrt{2}}$ and let c_n be a rational sequence tending to α and so we can construct a sequence b_n corresponding to some decimal expansion, and suppose this number is not in (a, b) then we move it to (a, b) by adding or subtracting some digits and this will not change the limit.

14. Suppose f is discontinuous at some point x. Then there exists $\epsilon > 0$ such that for all $\delta > 0$, there exists $|h| < \delta$ such that

$$|f(x + h) - f(x)| > \epsilon.$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$ so there exists $r_1, r_2 \in \mathbb{Q}$ with $f(x) - \epsilon < r_1 < f(x) < r_2 < f(x) + \epsilon$.

Now let $\delta_n = \frac{1}{n}$ and we obtain a corresponding h_n for each δ_n such that $|f(x+h_n) - f(x)| > \epsilon$. Then either $f(x+h_n) - f(x) > \epsilon$ in which case $f(x+h_n) > r_2$ or $f(x+h_n) - f(x) < -\epsilon$ in which case $f(x+h_n) < r_1$.

Thus, by the intermediate property, for each n we obtain a point x_n which lies between x and $x+h_n$ with either $f(x_n) = r_1$ or $f(x_n) = r_2$. As $x_n \to x$ so any subsequence of x_n will converge to x. Without loss of generality, we may assume that there is a subsequence of h_n, say $h_{j(n)}$ such that the corresponding $x_{j(n)}$ has the property $f(x_{j(n)}) = r_1$ (as there are only 2 values to pick so one of them must appear infinitely many times). Then $x_{j(n)} \to x$ but $f(x) \neq r_1$ which contradicts the fact that S_{r_1} is closed.