1A Analysis Example Sheet 3

zc231

IVT stands for intermediate value theorem and MVT stands for mean value theorem.

1. For any $\epsilon > 0$, pick $\delta < \epsilon$ so that if $|h| < \delta$ we have
$$\left|\frac{f(x+h) - f(x)}{h}\right| \leq h < \epsilon.$$
This shows that f is differentiable at any $x \in \mathbb{R}$ and $f'(x) = 0$ everywhere. Thus, by MVT, for any $x < y$,
$$f(x) - f(y) = (x-y)f'(z), \text{ for some } z \in (x, y)$$
which is zero and so $f(x) = f(y)$.

2. For $x \neq 0$ it is clear that the derivative is $2x\sin(1/x) - \cos(1/x)$. For $x = 0$, we compute
$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} x \sin(1/x) = 0$$
as $|\sin(1/x)|$ is bounded above by 1. So we conclude that $f'(x)$ is continuous at $x \neq 0$ and discontinuous at $x = 0$.

   For (ii) take for example $f(x) = x^{\frac{3}{2}}\sin(1/x)$ then $f'(x) = x^{\frac{1}{2}}\sin(1/x) - x^{-\frac{1}{2}}\cos(1/x)$ for $x \neq 0$ and is unbounded in $[-1, 1]$.

3. No. Consider firstly the bump function $f(x) = \exp(-\frac{1}{x^2(1-x^2)}), |x| < 1$ and $f(x) = 0$ if $|x| \geq 1$. This is smooth and $f(x) = o(x)$. Then consider another function which is basically differentiable but not twice differentiable. For example, let $g(x) = (x-5)^2$ for $x > 5$ and $g(x) = 0$ for $x \leq 5$. Then $g(x)$ is differentiable on (with $g'(5) = 0$) and $g'(x) = 0$ when $x \leq 5$ and $g'(x) = 2(x-5)$ if $x > 5$. But g is not twice differentiable as if $h < 0$ we have $\lim_{h\to 0} \frac{g'(5+h)-g'(5)}{h} = 0$ and if $h > 0$ we have $\lim_{h \to 0} \frac{g'(5+h)-g'(5)}{h} = 2$. Then we take $h(x) = f(x) + g(x)$ which has the property that $h(x) = o(x)$, $h(x)$ is differentiable but not twice differentiable.

4. $\log(1+x)$ is differentiable on $[-\epsilon, 1]$ for any $\epsilon < 1$ so MVT applies to any $a, b \in (-\epsilon, 1)$ and in particular, we apply MVT to 0 and a/n for $n > a$ (note in the case $a < 0$ as we only care about the limit as $n \to \infty$ so we start with n such that $|a/n| < 1/2$ and so we apply MVT on $[-1/2, 1]$ (so that $a/n, 0$ is contained in the interior).

   Then we have
$$\log\left(1 + \frac{a}{n}\right) = \frac{a}{n(1+z)}, \text{ for some } z \in (0, \frac{a}{n}).$$
Thus, $\left(1 + \frac{a}{n}\right)^n = e^a e^{\frac{1}{1+z}}$ and as $n \to \infty$, $z \to 0$ so the result follows because exp is a continuous function.

5. Consider $f(x) = e^x$ then $f(x)$ is clearly differentiable on $[-1, 1]$ so apply MVT at points 0 and $\frac{\log a}{n}$ where $n > \log a$ so we have

$$a^{\frac{1}{n}} - 1 = \frac{\log a}{n} e^z, \text{ for some } z \in (0, \frac{\log a}{n}).$$

Thus, $n(a^{\frac{1}{n}} - 1) = \log a e^z \to \log a$ as $n \to \infty$.

5* No. $f'(c+\theta h) \to f'(c)$ would imply f' is continuous at c which is not always true. For example, consider the function $f(x) = x^2 \sin(1/x)$ (where $f(0)$ is defined to be 0) and it is easy to check f is differentiable everywhere with $f'(0) = 0$ and $f'(x) = 2x\sin(1/x) - \cos(1/x)$ for $x \neq 0$. Now $f(1) - f(-1) = 0 = (1 - (-1))f'(0)$ but $f'(x)$ is not continuous at 0.

6. We prove this by induction. We claim that for all n, $f^{(n)}(0) = 0$ and $f^n(x)$ is of the form $p(1/x)\exp(-1/x^2)$ where $p(x)$ is polynomial in x. To prove this (include the case $n = 1$) we only need to prove the statement that: if $f(x) = p(1/x)\exp(-1/x^2), f(0) = 0$ then f is differentiable everywhere with $f'(0) = 0$. It is clear that f is differentiable at $x \neq 0$. For $x = 0$, consider

$$\left|\frac{f(h) - f(0)}{h}\right| = \left|p\left(\frac{1}{h}\right)\exp\left(-\frac{1}{h^2}\right)\right| = \left|p(t)\exp(-t^2)\right|,$$

where $t = 1/x$. By definition of e^x we see that $e^{t^2} > t^N$ for all $N \in \mathbb{N}$ and hence as $|t| \to \infty$, $|p(t)\exp(-t^2)| \to 0$ and so $f'(0) = 0$.

This shows that if we expand $f(x) = \exp(-1/x^2)$ in terms of Taylor series around 0 up to n-th term, then $f(x)$ is equal to the remainder term and in this case the remainder term does not converge to zero.

7. We calculate the limit of a_{n+1}/a_n in each case. The first one has $R = 3/2$ The second one is a power series in z^3 so the power series has $R = 2$ viewed as a power series in z^3 and so $R = \sqrt[3]{2}$ when it is viewed as a power series for z. Note this argument follows from the fact that each power series, if not converging everywhere, converges inside a circle of radius R so if $|z| > \sqrt[3]{2}$ then $|z^3| > 2$ then the series will not converge.

   For the third one, $a_{n+1}/a_n = (1 + 1/n)^n \to e$ as $n \to \infty$ so $R = 1/e$. For the last one we use the root test so we need to compute the limit of

$$n^{\frac{\sqrt{n}}{n}} = n^{\frac{1}{\sqrt{n}}} = (\sqrt{n}^{\frac{1}{\sqrt{n}}})^2.$$

   The limit of $\sqrt{n}^{\frac{1}{n}}$ is the same as the limit of $n^{\frac{1}{n}}$ as $n \to \infty$ (you can replace n by m^2 for the previous one if you like and let $m \to \infty$) and if $a_n = n^{\frac{1}{n}}$ then $\log a_n = \frac{\log n}{n} \to 0$ and so $a_n \to 1$. Thus, $R = 1$ in this case.

8. $\sec^2 x$. The function $\tan x$ satisfies the condition for inverse function theorem so the derivative of arctan exists (formally speaking, for any n we consider $\tan x$ on $[-\pi/2+1/n, \pi/2-1/n]$ and it is bijective with its image). By inverse function theorem, the derivative is

$$\frac{1}{\cos^2 \arctan x} = \frac{1}{1 + x^2}.$$

For all $|x| < 1$, since $\lim_{n\to\infty} nx^n \to 0$ so we conclude that there exists N such that $x^n < 1/n$ for all $n \geq N$ and so by comparison test with the sequence $\sum_{n=N}^{\infty} \frac{1}{n^2}$ we conclude that $g(x)$ converges absolutely on $|x| < 1$. Thus we can use term by term differentiation, so

$$g'(x) = 1 - x^2 + x^4 - \ldots = \frac{1}{1+x^2} = f'(x) \text{ where } f = \arctan x.$$

Thus $f'(x) - g'(x) = 0$ and so $f(x) - g(x)$ is a constant. Evaluate x at 0 we conclude that $f(x) = g(x)$.

9. Since the limit $\frac{f'(x)}{g'(x)}$ exists at $x = 0$ so if we have a sequence $x_n \to 0$ with $g'(x_n) = 0$ then $\frac{f'(x)}{g'(x)}$ is unbounded and hence the limit cannot exist and so there must be some interval $(0, a)$ such that g' does not vanish. Same argument works for the case when $a < 0$.

   Then by considering the function $F(u) = f(x)g(u) - g(x)f(u)$ we see that $F(0) = F(x) = 0$. Since f, g are both differentiable so is F and so by Rolle's theorem there exists y such that $F(y) = 0$ which gives $\frac{f'(y)}{g'(y)} = \frac{f(x)}{g(x)}$.

   Then let $x \to 0$ then $y \to 0$ and so the result for (ii) follows. To be more precise you may need to write down the detail for this in the exam. For example: for all $\epsilon > 0$, there exists δ such that for all $y < \delta$,

$$\left| \frac{f'(y)}{g'(y)} - l \right| < \epsilon$$

   and so for all $x < \delta$, we have

$$\left| \frac{f(x)}{g(x)} - l \right| = \left| \frac{f'(y)}{g'(y)} - l \right| < \epsilon.$$

   For the limit of $\frac{1-\cos(\sin(x))}{x^2}$ we apply L'hopital and we consider

$$\lim_{x\to 0} \frac{\sin(\sin(x))\cos(x)}{2x}$$

   and then apply again we have

$$\lim_{x\to 0} \frac{\cos(\sin(x))\cos^2(x) - \sin(\sin(x))\sin(x)}{2} = \frac{1}{2}.$$

10. As (a_n) is bounded let $a = \limsup a_n$ which is not $\pm\infty$. Let $b_n = \sup_{m\geq n} a_n$ then for each $k > 0$, there exists N_k such that $|b_{N_k} - a| < \frac{1}{2k}$. By definition of b_{N_k} there exists a_{n_k} with $n_k \geq N_k$ such that $|a_{n_k} - b_{N_k}| < \frac{1}{2k}$. Then by triangle inequality we have $|a_{n_k} - a| < \frac{1}{k}$. Take the subsequence a_{n_k} and we see $a_{n_k} - a \to 0$ (as for each k we have some n_k such that $|a_{n_k} - a| < \frac{1}{k}$) and so $a_{n_k} \to a$. This implies B-W.

11. As $a_n \geq 0$ for all n so it is clear that $p_n \geq 1 + a_1 + \ldots + a_n \geq s_n$. For all $x \geq 0$ we have $\log(1+x) \leq x$ and so

$$\log p_n = \sum_{k=1}^{n} \log(1 + a_i) \leq \sum_{k=1}^{n} a_i = s_n$$

   which implies that $p_n \leq e^{s_n}$.

   If p_n converges, then $s_n \leq p_n$ converges. Suppose s_n converges, then e^{s_n} also converges and hence p_n converges.

   The product converges as s_n converges (to 3/4). A direct computation shows that $p_n = 2n/(n+1)$ and hence $p_n \to 2$.

12* Define $g(x) = f(x) - f(0) - xf'(0) - x^2 f''(0)/2$ which is well-defined as $f''(0)$ exists and we have $g(0) = g'(0) = g''(0) = 0$. Further, let

$$\psi(x) = \frac{g(x) - g(0)}{x} = \phi(x) - f'(0) - xf''(0)/2$$

and so $\phi(x)$ is differentiable at x if and only if $\psi(x)$ is. Also, as $\phi(0) = f'(0)$ we have $\psi(0) = 0$. It is clear that $\psi(x)$ is differentiable at any $x \neq 0$. For $x = 0$, we need to compute

$$\lim_{h \to 0} \frac{\psi(h) - \psi(0)}{h} = \lim_{h \to 0} \frac{g(h)}{h^2}.$$

Then we can check it satisfies the condition for L'Hopital rule, (question 9), so we have

$$\lim_{h \to 0} \frac{g'(h)}{2h} = \lim_{h \to 0} \frac{g''(h)}{2} = 0$$

where we apply L'Hopital again in the second inequality above. Therefore, the limit exists and ψ is differentiable at 0 with $\psi'(0) = 0$ and so $\phi'(0) = \frac{f''(0)}{2}$.

This is clearly true for $x = 0$ as we proved $\phi'(0) = \frac{f''(0)}{2}$. For $x \neq 0$, we have

$$\phi'(x) = \frac{-xf'(x) + f(x) - f(0)}{x^2} = \frac{f''(\theta x)}{2}$$

by second order MVT. In case you don't know how to prove this, consider the function

$$t(h) = hf'(h) - f(h) + f(0) - \frac{h^2}{2} B$$

and for some constant B such that $t(x) = 0$ and since $t(0) = 0$ we have by Rolle's Theorem, some constant $|\theta x| < |x|$ such that $t'(\theta x) = 0$ which gives the desired constant B. Hence,

$$f(-1) + f(1) - 2f(0) = \phi(1) - \phi(-1) = \phi'(z) = f''(c)/2$$

by MVT and above where $z \in (-1, 1)$ and $c = \theta z$ for some $\theta \in (0, 1)$.

12. Let $g(x) = f(x) - zx$ then g is differentiable and hence continuous so it attains its minimum. $g'(a) < 0$ and $g'(b) > 0$ so these cannot be minimum because for $\epsilon < |g'(a)|$ then there exists $\delta > 0$ such that for all $0 < h < \delta$, we have

$$|g(a + h) - g(a) - g'(a)h| < h\epsilon$$

and so $g(a + h) - g(a) < g'(a)h + h\epsilon < 0$ (similar argument works for b). Therefore, the local minimum c is attained in (a, b) and by Fermat's theorem, $g'(c) = 0$.

The following is an example such that there does not exist a differentiable function F with $F' = f$. So if we find some function f which does not have intermediate property, then it is not the derivative of some function F because $F' = f$ must have the intermediate property. For example, let $f(x) = 0, x < 0$ and $f(x) = 1, x \geq 1$.

13. Let S be a non-empty set which has an upper bound. Let U be the sets of upper bounds for S and we define the indicator function by $f(x) = 1$ if $x \in U$ and -1 if $x \notin U$. We consider S with infinite cardinality or $|S| \geq 2$ (the case when S is finite is simple) so there exists x with $f(x) = 1$ and there exists x with $f(x) = -1$. But we have no x with $f(x) = 0$ so f is not

continuous by intermediate value property and let y be a point of discontinuity. We claim that y is the least upper bound for S.

Suppose we have $x \in S$ with $x > y$ then pick $\delta = x - y$ then if $z \in (y - \delta, y + \delta)$ we have $z < y + \delta = x$ so $z \notin U$ and so $f(z) = -1$ is constant on this interval and so f is continuous at y which is a contradiction. Therefore $y \geq x$ for all $x \in S$ and so $y \in U$.

Then for any $x \in U$ suppose $x < y$ then pick $\delta = y - x$ and so for each $z \in (y - \delta, y + \delta)$ we have $z > y - \delta = x$ and so $z \in U$. This implies $f(z) = 1$ is constant on this interval and hence f is continuous at y which is a contradiction. Therefore $y \leq x$ for all $x \in U$ and hence y is a least upper bound.

14. (i) $R = 1$ as $\lim_{n\to\infty}(n+1)/n = 1$. This is the same question as question 11 in sheet 1. Here is another proof. It is clear that the series diverges at $z = 1$. Let $|z| = 1, z \neq 1$. $z^{\frac{1}{2}} - z^{-\frac{1}{2}} = 2i\sin\theta$ then if S_n is the partial sum for the first n terms,

$$2i\sin\left(\frac{\theta}{2}\right)(S_n - S_m) = \sum_{k=m+1}^{n} \frac{z^{k+\frac{1}{2}} - z^{k-\frac{1}{2}}}{k} = \left[\sum_{k=m+2}^{n}\left(\frac{1}{k-1} - \frac{1}{k}\right) z^{k-\frac{1}{2}}\right] - \frac{z^{m+\frac{1}{2}}}{m+1} + a_n z^{n+\frac{1}{2}}.$$

Thus, the modulus of $2i\sin\left(\frac{\theta}{2}\right)$ is bounded above by

$$\sum_{k=m+2}^{n}\left(\frac{1}{k-1} - \frac{1}{k}\right) + \frac{1}{m+1} + \frac{1}{n} = 2\frac{1}{m+1}.$$

Hence we conclude that S_n is cauchy and so it converges for $|z| = 1$ with $z \neq 1$.

(ii) The series $v_i = \sum \frac{z^n}{n z_i^n}$ has radius of convergence 1 and converges at any $|z| = 1$ except $z = z_i$ (by using the same argument as in (i) or set $z' = z/z_i$). Then consider $v = v_1 + \cdots + v_m$. For $|z| < 1$ each v_i converges absolutely so v converges absolutely and for $|z| > 1$ it diverges because you can put v into the form $\sum \frac{z^n}{n} c$ for some constant c in terms of $z_1, \ldots, z_m$. Finally, if $z \neq z_1, \ldots, z_m$ then each v_i converges and so the sum converges and if $z = z_i$ for some i then v_j converges for $j \neq i$ but v_i diverges to ∞ so v diverges.

15. (i) We may assume g is the zero function as we can replace f by $f - g$ so now we assume that $r \geq 2$ and for each $i \leq r$, $f^{(j)}(x_i), j = 0, 1, \ldots, k_i - 1$. Thus by applying MVT we obtained $r - 1$ points, distinct from x_i at which f' vanishes. Together with the points from $x_i, i = 1, \ldots, r$ at which $f' = 0$ we have $b_1 = r - 1 + a_1$ points at which $f' = 0$ where a_1 is the number of $k_i \geq 2$. Similarly $b_2 = b_1 - 1 + a_2$ and in general $b_r = b_{r-1} - 1 + a_r$ where b_{r-1} is the number of points at which $f^{(r-1)} = 0$ and a_r is the number of $k_i \geq r + 1$. Setting $b_0 = r$ we have

$$b_{n-1} = a_{n-1} + \cdots + a_1 + b_0 - (n-1) = a_{n-1} + \cdots + a_1 + r - (n-1).$$

Note that $a_{n-1} + \cdots + a_1 + a_0 = n$ where $a_0 = r$ and hence $a_{n-1} + \cdots + a_1 = n - r$ and so $b_{n-1} = 1$. So the result follows and we need $r \geq 2$ so that $a_{n-1} = 0$ (so that the point we get is not any x_i).

(ii) The solution can be constructed by first principal and I will try to sketch how to do this. Consider the polynomials,

$$\begin{aligned} 1 \quad , \quad & x - x_1, \ldots, (x-x_1)^{k_1}, (x-x_1)^{k_1}(x-x_2), \ldots, (x-x_1)^{k_1}(x-x_2)^{k_2} \\ , \quad & (x-x_1)^{k_1}(x-x_2)^{k_2}(x-x_3), \ldots, (x-x_1)^{k_1}(x-x_2)^{k_2}\cdots(x-x_r)^{k_r-1} \end{aligned}$$

there are $1 + k_1 + \ldots + k_r - 1 = n$ of them so they form a basis for the polynomials of degree less than or equal to $n-1$. Note that we can replace 1 by $(x-x_1)^{k_1} \cdots (x-x_r)^{k_r}$ in the basis. Write $v_0 = 1, v_1 = x - x_1, v_2 = (x-x_1)^2, \ldots, v_{n-1} = (x-x_1)^{k_1} \cdots (x-x_r)^{k_r - 1}$. Let

$$p(x) = \sum_{s=1}^{n} a_s v_s.$$

Then $p(x_1) = f(x_1)$ implies $a_0 = f(x_1)$. Given $a_0, \ldots, a_s, s \leq k_1 - 2$, we can compute a_{s+1} by using the condition $p^{(s+1)}(x_1) = f^{(s+1)}(x_1)$ and we see $p^{(s+1)}(x_1) = (s+1)!a_{s+1}$ so we can compute a_{s+1}. Then for a_{k_1} we use the condition $p(x_2) = f(x_2)$ and we have $p(x_2) = a_{k_1}(x_2 - x_1)^{k_1}$. Then repeating this argument we see that we can determine each a_i and hence construct $p(x)$.