1A Analysis Example Sheet 4

zc231

1. $f(x) = x^2$. Consider the partition D with points $x_j = aj/n$, $1 \leq j \leq n$. Then

$$S(D, f) = \frac{a^3}{n^3} \sum_{j=1}^{n} j^2, s(D, f) = \frac{a^3}{n^3} \sum_{j=0}^{n-1} j^2$$

and $s(D, f) \leq \int_0^a x^2 dx \leq S(D, f)$. As $\sum_{j=1}^{n} j = (2n+1)(n+1)n/6$ so let $n \to \infty$ we conclude that $\int_0^a x^2 dx = a^3/3$.

2* A bounded function is integrable iff for all $\epsilon > 0$ there exists D such that $S(f, D) - s(f, D) < \epsilon$. It is clear that $\sin(1/x)$ is continuous on $[h, 1]$ for any $h > 0$ and hence integrable. Thus given any $\epsilon > 0$, let $h < \epsilon/4$ and pick D_1 a partition for $[h, 1]$ such that $S(f, D) - s(f, D) < \epsilon/2$. Let $D = D_1 \cup [0, h]$ and on $[0, h]$ the difference between inf and sup is at most 2 because $|\sin(1/x)| \leq 1$ and thus $h(\sup_{[0,h]} \sin(1/x) - \inf_{[0,h]} \sin(1/x)) \leq \epsilon/2$ and hence $S(D, f) - s(D, f) < \epsilon$. So it is integrable.

2. One example is to set up a function which is zero everywhere except we have a triangle with small values of bases and large values of heights around each integer. So formally, we take

$$f(x) = k^4(x - k + \frac{1}{k^3}), k - \frac{1}{k^3} < x < k, f(x) = -k^4(x - k - \frac{1}{k^3}), k < x < k + \frac{1}{k^3}, k \in \mathbb{N}$$

and $f(x) = 0$ otherwise. Then we see this is continuous and the integral

$$\int_0^\infty f(x)dx = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges and it is clear that $f(x)$ is unbounded.

3. Let $f(x) = 0, x < 1$ and $f(1) = 1$. Then it is clearly integrable (by the same argument as in the previous question) and $f(1) > 0$.

Suppose f is continuous now and if we have $f(x) = a > 0$ at some point x. f is integrable and $\int_0^1 f dx \geq s(f, D)$ for any partition D. Since f is continuous so there exists $\delta > 0$ such that for all $|h| < \delta$ we have

$$|f(x + h) - f(x)| < \frac{a}{2}$$

and in particular $f(x+h) > \frac{a}{2}$ for all $|h| < \delta$. Thus, take the partition D such that the interval containing x is $[x - \delta, x + \delta]$ and since $f \geq 0$ we have

$$s(f, D) \geq 2\delta \inf_{[x-\delta, x+\delta]} f(x) = a\delta > 0$$

and hence $\int_0^1 f dx \geq s(f, D) > 0$.

4. Without loss of generality we assume f is increasing. Let S be the set of discontinuities of f. For each $x \in S$ then there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $|h| < \delta$ such that $|f(x+h) - f(x)| > \epsilon$ and as f is increasing so $f(x+h) - f(x) > \epsilon$ if $h > 0$ and $f(x+h) - f(x) < -\epsilon$ if $h < 0$. We may assume $h > 0$. $\mathbb{Q}$ is dense in $\mathbb{R}$ so there exists $q \in \mathbb{Q}$ with $f(x) < q < f(x) + \epsilon$. Suppose q is also contained in $(f(z), f(z) + \epsilon_1)$ for some other $z \in S, z > x$ then we must have $f(z) < f(x)+\epsilon$ then we can pick δ small enough with $x+\delta < z$ and there exists $h < \delta$ so that $f(x+h) > f(x) + \epsilon > f(z)$ which contradicts the fact f is increasing. Thus we have an injection $S \to \mathbb{Q}$ and so S is countable.

   It is clear f converges for each x because $f \le \sum_{n=1}^{\infty} 2^{-n}$ which converges. f is increasing because for all $h > 0$, $x \ge x_n$ implies $x + h \ge x_n$ and since f is bounded and increasing so it is integrable.

   Fix x_N. Then it is clear that as f is increasing, for all $\delta > 0$ there exists $0 < h < \delta$ so that $f(x_N - h) + 2^{-N} \le f(x_N)$, i.e. $f(x_N) - f(x_N - h) \ge 2^{-N}$ and so it is discontinuous.

5* $f(0) = 0$ and apply MVT to an interval containing $[0, 1/2]$ we have

$$f(x) = f(x) - f(0) = \frac{-2cx}{1-c^2}, c \in (0, x).$$

Since $f(x) \le 0$ on $[0, 1/2]$ so we have $|f(x)| = \frac{2cx}{1-c^2} \le \frac{2x^2}{1-x^2}$ because $c/(1-c^2)$ is increasing in c and finally $(2x^2)/(1-x^2) \le 8x^2/3$ on $[0, 1/2]$.

We have

$$I_n = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \log\left(\frac{x}{n}\right) dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \log\left(1 + \frac{t}{n}\right) dt$$

where $x = t + n$. Now we split $[-1/2, 1/2]$ into $[-1/2, 0]$ and $[0, 1/2]$ and we have

$$\int_{-\frac{1}{2}}^{0} \log\left(1 + \frac{t}{n}\right) dt = \int_{0}^{\frac{1}{2}} \log\left(1 - \frac{t}{n}\right)$$

by $t \to -t$ and hence the resulting integral is $\int_0^{\frac{1}{2}} f(t/n)dt$. Then apply $|f(t/n)| \le 8t^2/3n^2$ we have $|I_n| \le 1/9n^2$.

Take the sum $\sum I_j$ we have

$$\left| \int_{\frac{1}{2}}^{n+\frac{1}{2}} \log x dx - \log n! \right| \le \sum_j \frac{1}{9j^2}.$$

We evaluate the left hand side, which gives $(n + 1/2)\log(n + 1/2) - \log(1/2) - n - \log n!$ and since $\log 2$ is a constant this gives

$$\left| \log\left(\frac{\left(n+\frac{1}{2}\right)^{\left(n+\frac{1}{2}\right)}}{e^{-n} n!} \right) \right| \le c$$

for some constant c.

In fact since each I_j is non-positive so we may drop the modulus and taking exponential we have

$$\left(\frac{n!}{\left(n+\frac{1}{2}\right)^{\left(n+\frac{1}{2}\right)} e^{-n}} \right) < c$$

for some constant c and as $n+1/2>n$ we have

$$\left(\frac{n!}{n^{\left(n+\frac{1}{2}\right)}e^{-n}}\right)<c$$

for some constant c. We can see directly from a computation of the ratio a_{n+1}/a_n to see the sequence is increasing (or by the fact that it is the exponential of $\sum|I_j|$) so any increasing sequence bounded above must be convergent.

5. We show that the integral exists and is zero and it suffices to prove that given any $\epsilon<0$ there exists D such that $S(f,D)<\epsilon$ (as $s(f,D)=0$ for any partition D). Given $\epsilon>0$, pick N such that $\frac{1}{N}<\frac{\epsilon}{2}$. There are only finitely many x with $f(x)>\frac{1}{N}$ say k of them. For each such x take an interval around x with length less than $\frac{\epsilon}{2k}$. Then by definition, as $\sup I\leq 1$ for any interval I we have

$$S(f,D)\leq\sum_{i\in I}L_i+\sum_{j\in J}\sup_j f(x)L_j$$

where L_i,L_j are the length of intervals and I contains intervals consisting of the k points with $f(x)>\frac{1}{N}$ and J contains all other intervals. Then for $i\in I$ we have $L_i<\frac{\epsilon}{2k}$ and we have k of them so the first sum is bounded above by $\frac{\epsilon}{2}$. For the second sum we have $f(x)<\frac{1}{N}$ for all x in the intervals $j\in J$ and hence the sum is at most $\frac{1}{N}<\frac{\epsilon}{2}$. Therefore $S(f,D)<\epsilon$.

6. Assume $\int_a^b f(x)dx=0$. Suppose the conclusion is not true then there exists a closed interval I of positive length and $\epsilon>0$ such that for all closed intervals $J\subset I$ of positive length we have $f(x)>\epsilon$ for some $x\in J$. Then take a partition D which contains I and then $s(f,D)$ will be at least ϵL_I where L_I is the length of I and so the integral will be at least ϵL_I which is a contradiction.

   Now if $f(x)>0$ and is integrable then the integral is ≥ 0. Suppose the integral is zero then the statement holds above. Firstly fix any subinterval $I_1=[c,d]$ there exists $x_1\in J_1\subset I_1$ such that $f(x)\leq 1$ for all $x\in J_1$. For $n\geq 2$ let $I_n=J_{n-1}$ and $\epsilon_n=\frac{1}{n}$ so we have $f(x)\leq\frac{1}{n}$ for all $x\in J_n$. Note each J_n has positive length. Then the intersection $J=\cap_n J_n$ is non-empty and pick $x\in J$ we have $f(x)\leq\frac{1}{n}$ for all n and so $f(x)=0$ which is a contradiction as we have $f(x)>0$ for all x.

7* The first part follows from integration by part. It is clear that $I_{2n+1}\leq I_{2n}$ because $0\leq\cos x\leq 1$ in $[0,\pi/2]$. Also, $(2n+1)I_{2n+1}=2nI_{2n-1}\geq 2nI_{2n}$ so we have the desired inequality.

We have $I_0=\pi/2$ and $I_1=1$. Therefore,

$$I_{2n}=\frac{(2n-1)(2n-3)\cdots 1}{(2n)(2n-2)\cdots 2}\frac{\pi}{2},I_{2n+1}=\frac{(2n)(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3}$$

and so

$$\frac{I_{2n+1}}{I_{2n}}=\frac{2}{\pi}\frac{(2n)^2\cdots 2^2}{(2n+1)(2n-1)^2\cdots 3^2}$$

which tends to 1 by the inequalities above which then gives the Wallis's product.

Let x_n be the n-th term. By previous question we know the sequence converges to l. Then the subsequence x_{2n} converges to the same limit and by limit operation we conclude that $(x_n)^2/x_{2n}\to l$. This gives

$$\frac{2^{2n}\sqrt{2}}{\sqrt{n}}\binom{2n}{n}^{-1}\to l.$$

Taking square of both sides we have

$$\frac{2^{4n}4}{2n}\binom{2n}{n}^{-2} \to l^2.$$

As

$$\frac{2^{4n}}{2n+1}\binom{2n}{n}^{-2} \to \frac{\pi}{2}$$

we have, taking quotient of the two,

$$4\frac{2n+1}{2n} \to l^2\frac{2}{\pi}.$$

But it is clear that the left hand side tends to 4 and so $l^2 = 2\pi$ which gives $l = \sqrt{2\pi}$.

7. (i) For all $x > 0$ we have $\sin x \leq x$ and hence

$$\int_0^\infty \sin^2\left(\frac{1}{x}\right) dx \leq \int_0^\infty \frac{1}{x^2} dx$$

which converges.

(ii) As $e^t > \frac{t^N}{N!}$ for all $N, t > 0$ therefore, $\frac{1}{e^t} < \frac{N!}{t^N}$ and now we pick N such that $qN > p+2$ so that $e(-x^q) < \frac{N!}{x^{p+2}}$ and so $x^p e(-x^q) < \frac{N!}{x^2}$. Hence the integral converges.

(iii) For each n we consider the partial integral

$$\int_{\sqrt{2n\pi}}^{\sqrt{(2n+1)\pi}} \sin(x^2)dx + \int_{\sqrt{(2n+1)\pi}}^{\sqrt{(2n+2)\pi}} \sin(x^2)dx$$

and set $y = x^2$ so $\sin(x^2)dx = \frac{\sin(y)}{2\sqrt{y}}dy$ and for the second interval we further set $z = y - \pi$ and so if we tidy up the expression we have

$$\int_{2n\pi}^{(2n+1)\pi} \frac{\sin(y)}{2\sqrt{y}} - \frac{\sin(y)}{2\sqrt{y+\pi}}dy > 0.$$

Finally when $n \to \infty$ the integral

$$\int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \sin(x^2)dx \leq \sqrt{(n+1)\pi} - \sqrt{n\pi} = \frac{1}{\sqrt{(n+1)\pi} + \sqrt{n\pi}} \to 0$$

and so by alternating test the integral converges.

8. By definition of integration we take the partition D with intervals $[n, n+1], [n+1, n+2], \ldots, [2n-1, 2n]$ for $f(x) = 1/x$ and $g(x) = 1/(x+1)$. Then

$$S(D, g) = s(D, f) = \frac{1}{n} + \ldots + \frac{1}{2n}$$

and so we have

$$\int_n^{2n} dx/(x+1) \leq S(D, g) = s(D, f) \leq \int_n^{2n} dx/x$$

and so $\log(\frac{2n+1}{n+1}) \leq \sum_{i=1}^n \frac{1}{n+i} \leq \log 2$. Then take the limit as $n \to \infty$.

The second series converges by alternating test and so the limit exists and is the same as the limit of

$$\frac{1}{2n+1} - \frac{1}{2n+2} + \cdots + \frac{-1}{4n}$$

because this is a subsequence of the original one (we replace n by $2n$).

By the first part we conclude the limit of the subsequence

$$a_n = \frac{1}{2n+1} + \cdots + \frac{1}{4n} \to \log 2$$

and

$$b_n = \frac{1}{2n+2} + \frac{1}{2n+4} + \cdots + \frac{1}{4n} \to \frac{\log 2}{2}.$$

Take $a_n - 2b_n$ we have the required sequence above and the limit is 0.

9. True. Suppose not, say $f(y) \neq 0$ and we may assume $f(y) = t > 0$. Then pick $\epsilon = t/2$, there exists δ so that for all $|h| < \delta$, $|f(y+h) - t| < \epsilon$ and hence $f(y+h) > t/2$. We now construct g with $g(x) = 0$ if $|x-y| \geq \delta$ (pick δ small enough so that $|x-y| > \delta$ is still in $[a,b]$) then $g(a) = g(b) = 0$. For $|x-y| \leq \delta$, let $g(x) = x + \delta$ if $y - \delta < y$ and $g(x) = -(y+\delta)(x-y-\delta)/\delta$, then $g(x)$ is continuous and $g(x) > 0$ for $|x-y| < \delta$. Therefore,

$$\int_a^b f(x)g(x)dx = \int_{y-\delta}^{y+\delta} f(x)g(x)dx > 0.$$

10. By definition

$$g(x) = (x-1)\int_0^x f(t)tdt + x\int_x^1 f(t)(t-1)dt = (x-1)f_1(x) + xf_2(x).$$

By fundamental theorem of calculus, for each $x \in (0,1)$, $f_1'(x) = xf(x)$ and $f_2'(x) = -f(x)(x-1)$. Thus, we have

$$g'(x) = \int_0^x f(t)tdt + \int_x^1 f(t)(t-1)dt + (x-1)xf(x) - xf(x)(x-1) = \int_0^x f(t)tdt + \int_x^1 f(t)(t-1)dt.$$

We then again apply fundamental theorem of calculus so that $g''(x) = f(x)x - f(x)(x-1) = f(x)$.

11* Fix $x > 0$ and we have

$$\left|\frac{f(x)}{x}\right| = \lim_{h\to 0}\left|\frac{f(x)-f(h)}{x-h}\right| = |f'(z)| \leq M$$

for some z between h and x. Thus, we have $|f(x)| \leq Mx$ for all $x > 0$ and therefore

$$\left|\int_0^1 f(x)dx\right| \leq \int_0^1 |f(x)|dx \leq \int_0^1 Mxdx = M/2.$$

By the same method above we have for each x, $|f(x)| \leq M(1-x)$. Therefore,

$$\begin{aligned}\left|\int_0^1 f(x)dx\right| &\leq \int_0^1 |f(x)|dx = \int_0^y |f(x)|dx + \int_y^1 |f(x)|dx \\ &\leq \int_0^y Mxdx + \int_y^1 M(1-x)dx = \frac{M}{2} - My + My^2\end{aligned}$$

for any $y \in (0,1)$. Pick $y = 1/2$.

Suppose now $|f'(x)| \le Mx$. As $f'(x)$ is continuous so we apply fundamental theorem of calculus, and we have

$$|f(x)| = \left|\int_0^x f'(t)dt\right| \le \int_0^x |f'(x)|\, dx \le \frac{Mx^2}{2}.$$

Therefore, $\left|\int_0^1 f(x)\right| dx \le M/6$.

11. By definition for any a, b we have

$$L(a) = \int_1^a \frac{dt}{t} = \int_b^{ab} \frac{dz}{z}$$

where we used the substitution $z = bt$ and thus

$$L(a) + L(b) = \int_1^{ab} \frac{dt}{t} = L(ab).$$

Define e to be the least positive number such that $L(e) = 1$. In fact this is unique as L is increasing. Also by fundamental theorem of calculus we conclude that L is continuous and $L(x) \to \infty$ as $x \to \infty$ so by IVT there exists e with $L(e) = 1$ (as $L(1) = 0$).

12. The first part follows by using integration by part twice and direct computation. Then the claim is clearly true for $n = 0, 1$ by direct computation. Then by induction,

$$\begin{aligned} \theta^{2n+1} I_n &= \theta^{2n-1}\theta^2 I_n = \theta^{2n-1}(2n(2n-1)I_{n-1} - 4n(n-1)I_{n-2}) \\ &= 2n(2n-1)(n-1)!(P_{n-1}\sin\theta + Q_{n-1}\cos\theta) \\ &\quad -\theta^2 4n(n-1)(n-2)!(P_{n-2}\sin\theta + Q_{n-2}\cos\theta) \end{aligned}$$

which has the desired form (as $n! = n(n-1)! = n(n-1)(n-2)!$ and the degree of P_n, Q_n is bounded by 2 plus the degree of P_{n-2}, Q_{n-2}).

As $1 - x^2 \le 1$ and $|\cos\theta x| \le 1$ we conclude that $|I_n| \le 2$. Suppose π is rational, say $\pi = \frac{p}{q}$ for some coprime integers p, q. Then we substitute $\theta = \pi$ into the equation above and rearrange the equation we have,

$$\frac{p^{2n+1}}{qn!} I_n(\pi) = -Q_n(\pi) q^{2n}.$$

As Q_n has degree at most $2n$ so the right hand side is an integer. But $|I_n(\pi)| \le 2$ so we may pick n large enough so that the left hand side has modulus less than 1, which is always possible because

$$\lim_{n\to\infty} \frac{p^{2n+1}}{n!} = 0.$$

This gives a contradiction.

13. $g(x) = x\sin(1/x)$ is continuous and hence integrable. Suppose $g(x) = f_1 - f_2$ where f_1, f_2 are increasing. Then for any $x_0 < x_1 < \cdots < x_n$ we have

$$|g(x_j) - g(x_{j-1})| = |f_1(x_j) - f_1(x_{j-1}) - f_2(x_j) + f_2(x_{j-1})| \le |f_1(x_j) - f_1(x_{j-1})| + |f_2(x_j) - f_2(x_{j+1})|.$$

Since f_1, f_2 both increasing, so

$$\begin{aligned} \sum_{j=1}^n |g(x_j) - g(x_{j-1})| &\le \sum_{j=1}^n (f_1(x_j) - f_1(x_{j-1})) + \sum_{j=1}^n (f_2(x_j) - f_2(x_{j-1})) \\ &= f_1(x_n) - f_1(x_0) + f_2(x_n) - f_2(x_0) \end{aligned}$$

and so in particular this is bounded (only depends on x_0 and x_n, independent of n).

Then we consider the dissection $x_0 = 0, x_j = \frac{2}{(2n-2j-1)\pi}, 1 \leq j \leq n-1, x_n = 1$ then for each $2 \leq j \leq n-1$,

$$\frac{\pi}{2}|g(x_j) - g(x_{j-1})| = \frac{1}{(2n-2j-1)} + \frac{1}{2n-2j+1}.$$

Thus $\frac{\pi}{2}\sum_{j=2}^{n-1}|g(x_j) - g(x_{j-1})| \geq 2\frac{1}{3} + 2\frac{1}{5} + \cdots + 2\frac{1}{2n-5} + \frac{1}{2n-3}$ which is unbounded and this gives a contradiction.

14. Suppose not then we have an interval $[a, b] \subset [0, 1]$ on which f is nowhere continuous. Since f is integrable on $[a, b]$, then given any $\epsilon > 0$ there exists a partition D with $S(f, D) - s(f, D) < \epsilon(b-a)$. For each interval I in D, let $M(I) = \sup_I f(x)$,$m(I) = \inf_I f(x)$ then pick $I \in D$ the interval with $M(I) - m(I)$ minimal. Then as $\sum_i x_i - x_{i-1} = b - a$,

$$\begin{aligned}(b-a)(M(I) - m(I)) &= \sum_i (M(I) - m(I))(x_i - x_{i-1}) \\ &\leq \sum_i (M(I_i) - m(I_i)(x_i - x_{i-1}) = S(f, D) - s(f, D) < \epsilon(b-a).\end{aligned}$$

In particular, $M(I) - m(I) < \epsilon$.

Now we vary ϵ. Let $\epsilon_n = 1/n$. Let $I_1 \subset [0, 1]$ such that $M(I_1) - m(I_1) < 1$. For $n \geq 2$, as f is integrable on $[0, 1]$ so it is integrable on I_{n-1}. Thus, using the above argument we pick a subinterval $I_n \subset I_{n-1}$ such that $M(I_n) - m(I_n) < 1/n$ and if I_n is not strictly contained in I_{n-1} then we just shrink I_n so that the endpoints of I_n are not endpoints of I_{n-1}. Thus, we have a sequence of nested closed intervals $I_1 \supset I_2 \supset \cdots$, and so the intersection is non-empty. Let x be any point in the intersection.

We show that f is continuous at x. For all $\epsilon > 0$, there exists n with $1/n < \epsilon$, and so pick δ small enough such that $[x - \delta, x + \delta]$ is contained in I_n, then for all $|h| < \delta$ we have

$$|f(x + h) - f(x)| \leq M(I_n) - m(I_n) < \frac{1}{n} < \epsilon.$$

Hence x is a point of continuity of f. But we assumed that f is nowhere continuous on $[a, b]$ which is a contradiction.