PartIA Analysis

zc231

Each question will be labeled in the form $\alpha, \beta\gamma$ where $\alpha \in \{1, 2, 3, 4\}$ represents the paper number, $\beta\gamma$ represents the question number in that paper. For example, 1,11G means question 11G in paper 1. I will omit the proofs in the notes or book work. The solutions provided might not be the best ways to solve the problems and if you find any mistakes or if you have any elegant ways of solving some of the problems please email me at zc231@cam.ac.uk.

2009

1,3F $a_n = \frac{n}{n+\sqrt{n^2-n}} = \frac{1}{1+\sqrt{1-1/n}} \to \frac{1}{2}$. Let $r_n = \sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n} \to \frac{1}{2}$. Then

$$\cos(2\pi(\sqrt{n^2+n})) = \cos(2\pi r_n + 2\pi n) = \cos(2\pi r_n)\cos(2\pi n) - \sin(2\pi r_n)\sin(2\pi n) = \cos(2\pi r_n)$$

and $\cos(2\pi r_n) \to \cos(\pi) = -1$ (you probably need to state that cos is continuous). Then use the fact

$$2\cos^2 x - 1 = \cos 2x$$

so $b_n \to ((-1) + 1)/2 = 0$.

1,4F The first part is book work. Let $a_n = (z^2 - 1)$.

1,9F (a) If $\sin(x) = 1$ then clearly the series diverges. If $\sin(x) = -1$ then the series converges because it is alternating and decreasing. But it does not converge absolutely because the modulus of each term has the form $\left|\frac{3(-1)^n+1}{n}\right| \geq \frac{2}{n}$.
Let $r = \sin x$ and $a_n = \frac{3+r^n}{n}$ then

$$\sum_n |a_n| \leq \sum_n \frac{3|r|^n}{n} + \sum_n \frac{|r|^{2n}}{n}$$

which converges for any $|r| < 1$ so for any other values of x it converges absolutely.

(b) For all value of x, the series converges because it is alternating and decreasing. If $\sin x \neq \pm 1$ then the series converges absolutely because

$$\sum_{n=1}^{\infty} \frac{r^n}{\sqrt{n}} < \sum_{n=1}^{\infty} r^n.$$

If $\sin x = \pm 1$ then it does not converges absolutely because

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \sum_{n=1}^{\infty} \frac{1}{n}.$$

(Well I guess in the exam you probably need to explain why $\sum \frac{1}{n}$ diverges (which is standard), and by the same argument you can show $\sum \frac{1}{\sqrt{n}}$ diverges so may be there is no need to use comparison).

(c) $\sin(0.99x) \in [-1,1]$ and so $\sin(0.99x) \in [-\pi/2, \pi/2]$. Therefore, if $\sin(0.99x) > 0$ then $a_n > 0$ for all n and if $\sin(0.99x) < 0$ then $a_n < 0$ for all n. So in this case, if the series converges then it is absolutely convergent.

Firstly assume $\sin(0.99x) > 0$. Since $\sin x \le x$ for $x \ge 0$ so $\sin(0.99\sin(x)) \le 0.99x$ and by induction, it is clear that $a_n \le 0.99^n x$ and thus by comparison this converges for all x with $\sin(0.99x) > 0$ (as x is fixed so basically it is a sum of geometric sequence). Now suppose $\sin(0.99x) < 0$ then $a_n = -b_n$ where $b_n = \sin(0.99\sin(\cdots(\sin(-0.99x))))$ and $\sin(-0.99x) > 0$ so $b_n \le 0.99^n(-x)$ and so $\sum_n b_n$ converges for all x with $\sin(0.99x) < 0$ and hence $\sum_n a_n$ converges. Therefore, we conclude that the series converges absolutely for all x.

1,10D Let $d(x) = (x-a)^2 + (f(x)-b)^2$ which is the square of the distance from a point $(x, f(x))$ to $P = (a,b)$. Since $f(x)$ is continuous, so is $d(x)$. We firstly show that $d(x)$ has a minimum. Clearly $d(x)$ is bounded below by 0 so $d(x)$ has a greatest lower bound, say M and so there exists a sequence $x_n \in \mathbb{R}$ with $d(x_n) \to M$. Also $d(x) \ge (x-a)^2 > 2M$ whenever $|x| > N$ for some large N. Therefore, for n large enough we have $|x_n| \ge N$. Then by B-W we have a convergent subsequence $x_{n_j} \to y$ and $d(x_{n_j}) \to d(y)$ because d is continuous and $x \in \mathbb{R}$ as $\mathbb{R}$ is complete. But $d(x_n) \to M$ so we conclude that $d(y) = M$ and so the greatest lower bound is attained. So $d(x) \ge M$ for all $x \in \mathbb{R}$.

Given any number $r \in \mathbb{R}$, $r > M$, pick x with $d(x) > r$ (which is always possible as $d(x) \ge (x-a)^2)$ then by IVT we conclude that there exists c between x and y such that $d(c) = r$. Since r is arbitrary so we conclude that the image of $d(x)$ is an interval. Hence the distance function is an interval (because if $d'(x)^2 = d(x)$ then $d'(x) \ge \alpha$ if and only if $d(x) \ge \alpha^2$).

1,11I $\phi(a) = \phi(b) = -g(b)f(a) + f(b)g(a)$ and so the result follows by Rolle's theorem. Take a, b with $0 < a < b$ and $g(a) \ne g(b)$ and we have

$$\frac{f(a)-f(b)}{g(a)-g(b)} = \frac{f'(c)}{g'(c)}, a < c < b.$$

As $\frac{f'(x)}{g'(x)} \to l$ so for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$l - \frac{\epsilon}{2} < \frac{f'(x)}{g'(x)} < l + \frac{\epsilon}{2}, 0 < x < \delta.$$

Thus pick $b \le \delta$ and $0 < a < b \le \delta$ we have, from above that

$$l - \frac{\epsilon}{2} < \frac{f(a)-f(b)}{g(a)-g(b)} = \frac{f'(c)}{g'(c)} < l + \frac{\epsilon}{2}, 0 < a < c < b \le \delta.$$

Now let $a \to 0$, so we have

$$l - \epsilon < l - \frac{\epsilon}{2} \le \frac{f(b)}{g(b)} \le l + \frac{\epsilon}{2} < l + \epsilon, 0 < b \le \delta.$$

As ϵ is arbitrary, this shows that $\lim_{b\to 0} \frac{f(b)}{g(b)} = l$.

1,12E For the second part, let $\sup f - \inf f = M$. For all $\epsilon > 0$, pick $\delta < \frac{\epsilon}{2M}$ and take the a_i, b_i's by assumption. Then on each $[a_i, b_i]$, since f is integrable so we pick a partition D_i with $S(f, D_i) - s(f, D_i) < \frac{\epsilon}{2n}$. On $[b_i, a_{i+1}]$, we have the contribution

$$S(f, [b_i, a_{i+1}] - s(f, [b_i, a_{i+1}]) \le M.$$

and since $\sum_i (b_i - a_i) \geq 1 - \delta$ so $\sum_i (a_{i+1} - b_i) \leq \delta$ so the total contribution is less than $M\delta$. Therefore, let $D = \cup_i D_i \cup_i [b_i, a_{i+1}]$ we have

$$S(f, D) - s(f, D) < n\frac{\epsilon}{2n} + M\delta < \epsilon.$$

Fix n and let $a_i = 10^{-n}i + 100^{-n}$. Take $D = \{a_i : i = 0, 1, \ldots, 10^n - 1\} \cup \{0, 1\}$. Then there are exactly 2^n intervals for which f is non-zero (to see this, consider $b_i = 10^{-n}i$ then there are 2^n points of b_i are non-zero). Then $S(f, D) = 2^n 10^{-n} = 5^{-n}$. Therefore, for each $\epsilon > 0$ pick n with $5^{-n} < \epsilon$ we have $S(f, D) < \epsilon$ and hence f is integrable and the integral is 0.

2010

1,3D The sum $\sum_n a_n z^n$ diverges for $|z| > R$. Suppose $|z| = S > R$, and $|a_n z^n|$ is bounded by M, then $|a_n| \le \frac{M}{S^n}$. Now pick r with $R < r < S$, then for $|z| = r$, we have

$$\sum_n |a_n z^n| \le \sum_n M \frac{r^n}{S^n}$$

which converges, and so R is not the radius of convergence, which is a contradiction. The radius of convergence is 1 by ratio test.

1,4E $1 + 2 + \cdots + n = n(n+1)/2$ and so the limit for the first one is $1/2$. For the second one, let $a_n^n = n$ so $\log a_n = \frac{\log n}{n} \to 0$ so $a_n \to 1$. For the last one, we have $b(1 + \frac{a^n}{b^n})^{\frac{1}{n}} \to b$ because $\frac{a}{b} \le 1$ and $c^{\frac{1}{n}} \to 1$ for any constant c.

1,9E (a) By ratio test, $\frac{a_{n+1}}{a_n} = \frac{n^n}{(n+1)^n} \to \frac{1}{e}$ so the series converges.

(b) By condensation test we consider $\sum a_n$ with

$$a_n = \frac{2^n}{2^n + n^2 \log^2 2} = \frac{1}{1 + \log^2 2 \frac{n}{2^n}}$$

but $a_n \to 1$ as $n \to \infty$ so the series diverges (if it converges then $a_n \to 0$).

(c) Converges by alternating test.

(d) Consider when n is odd then

$$a_n + a_{n+1} = -\frac{1}{n} + \frac{1}{3(n+1)} < -\frac{1}{n} + \frac{1}{3n} = \frac{-2}{3n} < -\frac{1}{3n} - \frac{1}{3(n+1)}.$$

Thus $|3\sum_{n=1}^{N} a_n| > \sum_{n=1}^{N} \frac{1}{n}$ whenever N is even and the sum is unbounded so it diverges.

1,10F Taylor series with Lagrange remainder

$$f(a+h) = f(a) + hf'(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a + \theta h), \theta \in (0,1).$$

Since $e'(x) = e(x)$ and $e(x)$ is differentiable, by induction we conclude that $e(x)$ is infinitely differentiable and $e^{(n)}(x) = e(x)$ Pick $a = 0, h = x$ we have

$$e(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} e(\theta_n x), \theta_n \in (0,1)$$

and this is true for any n. Now for each x, e is differentiable and hence continuous, so let M be the maximum of $|e|$ on $[0,x]$ and hence $|e(\theta_n x)| \le M$ whatever θ_n is (as long as $\theta_n < 1$). Then for all $\epsilon > 0$, there exists N such that for all $n > N$,

$$\left|\frac{x^n}{n!}\right| M \le \epsilon$$

because $\frac{x^n}{n!} \to 0$ as x is fixed. Therefore, the series converges to $e(x)$ and so

$$e(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

1,11D The first two are book work. The third part is integral test, and use that for the last part, we consider, for $\alpha < 0, \alpha \neq -1$ (clearly if $\alpha \geq 0$ then a_n is unbounded hence the series diverges) $\int_1^\infty x^\alpha dx$ which diverges for $\alpha \geq -1$ and converges otherwise.

1,12F f is differentiable at 0 and $f'(0) = 0$ because by definition

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = x\text{sign}(x)\left|\cos\frac{\pi}{x}\right| = \left|x\cos\frac{\pi}{x}\right| \leq |x| \to 0$$

where we write $x\text{sign}(x) = |x|$ for $x \neq 0$.

The idea of the second part is basically to show $|\cos x|$ is not differentiable at any x with $\cos x = 0$ because clearly x^2 and $\text{sign}(x)$ are both differentiable at $x \neq 0$(this is just the intuition). Therefore, we consider $x = \frac{2}{2n+1}$ and let $h = \frac{2}{2n+1}\frac{-1}{m+1}$, then $x + h = \frac{2}{2n+1}\frac{m}{m+1}$ and $f(x) = 0$. Also pick $|m|$ large enough so that $x + h > 0$, and we assume n is even (this is not necessary but it makes the situation simple).

$$f(x+h) = (x+h)^2\left|\cos\left(\frac{2n+1}{2}\pi\frac{m+1}{m}\right)\right| = (x+h)^2\left|\cos\left(\frac{2n+1}{2}\pi(1+\frac{1}{m})\right)\right|$$

which then gives

$$(x+h)^2\left|\cos\left(\frac{2n+1}{2}\pi + \frac{2n+1}{2m}\pi\right)\right| = (x+h)^2\left|\sin\left(\frac{2n+1}{2m}\pi\right)\right|.$$

Therefore,

$$\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{m\to\pm\infty}(x+h)^2\left|\sin\left(\frac{2n+1}{2m}\pi\right)\right|\frac{2n+1}{2}(-m-1)$$

which we will rewrite as (and use $x + h = \frac{2}{2n+1}\frac{m}{m+1}$)

$$\lim_{m\to\pm\infty}(x+h)^2\left|\frac{\sin\left(\frac{2n+1}{2m}\pi\right)}{\frac{(2n+1)\pi}{2m}}\right|\frac{(2n+1)^2\pi}{4|m|}(-m-1) = \frac{\pi m^2}{(m+1)^2}\frac{(-m-1)}{|m|}.$$

where we used $\lim_{x\to 0}\frac{\sin x}{x} = 1$. We see the above limit is $-\pi$ if $m \to -\infty$ and is π if $m \to +\infty$ so the limit does not exist as $h \to 0$ (so the idea is similar to how you show $|x|$ is not differentiable at 0). As n is arbitrary this completes the second part. I think in the exam you can just explain this in the sense that $|\cos\frac{\pi}{x}|$ is not differentiable at $x = \frac{2}{2n+1}$.

Now if the limit exists at $x \neq 0$, which means (you can always drop the sign function by considering $x > 0$ or $x < 0$, because as $h \to 0$ with fixed x, $\text{sign}(x+h)$=$\text{sign}(x)$)

$$\lim_{h\to 0}\frac{x^2\left|\cos\frac{\pi}{x}\right| - (x+h)^2\left|\cos\frac{\pi}{x+h}\right|}{h} = \lim_{h\to 0} x^2\frac{\left|\cos\frac{\pi}{x}\right| - \left|\cos\frac{\pi}{x+h}\right|}{h} - 2x\left|\cos\frac{\pi}{x+h}\right| - h\left|\cos\frac{\pi}{x+h}\right|$$

exists and since the limit of the last term exists (which is 0) so the limit of the above is the same as the limit of the first two terms. Then it is clearly bounded because if you expand using triangle inequality then the second term is bounded by $2|x|$ and the first term is also bounded because at the point where the function is differentiable we must have $\cos\frac{\pi}{x} \neq 0$ and so for h small enough we conclude $\cos\frac{\pi}{x}$ and $\cos\frac{\pi}{x+h}$ have the same sign and so the first term is just $\pm x^2\frac{d\cos\frac{\pi}{x}}{dx}$. Hence if you take any finite interval I, this will be bounded by some constant C depending on I.

2011

1,3F (b): $a_n = n$ (c): $a_n = 0$ if n is odd and $a_n = n$ if n is even. (d): $a_{4n} = 2 + \frac{1}{4n}, a_{4n+1} = 2 + \frac{1}{4n+1}, a_{4n+2} = -\frac{1}{4n+2}, a_{4n+3} = -\frac{1}{4n+3}$. Therefore $c = 2$ or 0.

1,4D (i): $R = 1$ (ii): By root test we consider

$$(n^{n^{(1/3)}})^{(1/n)} = n^{n^{-(2/3)}} \to 1$$

so $R = 1$.

1,9F Converge: $a_n = \frac{1}{n^2}$. Diverge: $a_n = \frac{1}{n}$. For the second part, if you want to use the integral method (compare the area etc.) then I think you have to justify lots of properties in integration so here the method I suggest is: take $x = \frac{k-1}{k}$ for $k > 1$. Then we have $-\frac{k-1}{k} > \log\frac{1}{k}$ for all $k > 1$. Now $\frac{1}{k} = 1 - \frac{k-1}{k}$ for all $k > 1$. Thus, we have

$$\sum_{k=1}^{n-1}\frac{1}{k} = 1 + \sum_{k=2}^{n-1}(1 - \frac{k-1}{k}) \geq n - 1 + \sum_{k=2}^{n-1}\log\frac{1}{k} = n - 1 - \log(n-1)!.$$

Further, for $n \geq 2$, $n - 1 \geq \log n!$ (as $e^{n-1} \geq n$) and hence the result follows.

We can ignore the first several terms so we start with n large enough, say $n = N$ so that $\frac{c}{n} < 1$ and the condition holds then as $\frac{a_{n+1}}{a_n} < 1 - \frac{c}{n}$ we have

$$\log\frac{a_{n+1}}{a_n} < \log(1 - \frac{c}{n}) < -\frac{c}{n}$$

and now pick N large enough so that for all $n > N$, $e^{-\frac{c}{n}} < \frac{1}{2}$ so

$$a_{n+1} < a_n e^{-\frac{c}{n}} < \frac{1}{2}a_n.$$

Therefore, the series converges by comparison test.

1,10E If $f(0) = 0$ or $f(1) = 1$ then we are done. If not we have $f(0) > 0$ and $f(1) < 1$. Let $g(x) = f(x) - x$ so $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$ so by IVT there exists c such that $g(c) = 0$ so $f(c) = c$.

(i) No. For example $f(x) = x^2$ and $x^2 = x$ if and only if $x = 0, 1$.

(ii) No. For example, $f(x) = e^x$. For $x \leq 0$, $e^x > 0$ so $e^x \neq x$. For $x > 0$, $e^x = 1 + x + x^2/2 + \cdots > 1 + x$ so $e^x \neq x$.

(iii) No. For example $f(0) = 1, f(1) = 0$ and $f(x) = x^2$ for $0 < x < 1$.

(iv) Yes. For example, $f(x) = 1 - 2x$, $0 \leq x \leq \frac{1}{2}$, $f(x) = 3(x - \frac{1}{2})$, $\frac{1}{2} \leq x \leq \frac{3}{4}$ and $f(x) = -3(x-1)$, $\frac{3}{4} \leq x \leq 1$ then f is continuous, $f(0) = 1, f(1) = 0$ and $\frac{1}{3}, \frac{3}{4}$ are the only fixed points.

1,11E For the first one the only point of continuity is 0, for the others, say $a \neq 0$, if $a \notin \mathbb{Q}$ then for all $\delta > 0$, pick $h < \delta$ with $a + h \in \mathbb{Q}$, and $|f(a) - f(a+h)| = |2a - h| > |a|$ (with h small) and the case when $a \in \mathbb{Q}$ is similar. But f is not differentiable at 0 because let $a_n \in \mathbb{Q}$, $b_n \notin \mathbb{Q}$ be two sequences tending to 0, but

$$\frac{f(a_n) - f(0)}{a_n} = 1, \frac{f(b_n) - f(0)}{b_n} = -1$$

so the limit of $\frac{f(h)-f(0)}{h}$ as $h \to 0$ does not exist.

For the second one, it is continuous at every x and differentiable at $x \neq 0$. To check it is continuous at $x = 0$, by definition

$$\lim_{x \to 0} |f(x) - f(0)| = \lim_{x \to 0} |x \sin(1/x)| \leq \lim_{x \to 0} |x| = 0.$$

To see it is not differentiable at $x = 0$, we have

$$\frac{f(x) - f(0)}{x} = \sin(1/x)$$

which does not have a limit as $x \to 0$.

Take $f(x) = x^2$ if $x \in \mathbb{Q}$ and $f(x) = -x^2$ if $x \notin \mathbb{Q}$ then $f'(0) = 0$. By a similar argument as in (i) we see $f(x)$ is only continuous at 0 (hence not differentiable at any other point).

For the last part consider $f(x) = \left|x^2 \sin \frac{\pi}{x}\right|$ for $0 < x \leq \frac{1}{2}$, $f(0) = 0$ and $f(x) = x^2 \sin \frac{\pi}{x}$ for $x > \frac{1}{2}$. It is clear that this is not differentiable at $x = 1/2, 1/3, \ldots$ and $f'(0) = 0$. Outside the interval $[0, 1/2]$, $f(x) = x^2 \sin \frac{\pi}{x}$ which is clearly differentiable.

1,12D Let $f(x) = 0, 0 \leq x \leq 1/2$ and $f(x) = 1, 1/2 < x \leq 1$ then $f(x)$ is integrable. But $F(x) = 0$ if $x \leq 1/2$ and $F(x) = (x - 1/2)$ for $x > 1/2$ it is not differentiable because it is not even continuous at $x = 1/2$.

No. Let $f(x) = x^2 \sin(1/x^2), x \neq 0$ and $f(0) = 0$. Then $f(x)$ is differentiable everywhere and $g(0) = f'(0) = 0$,$g(x) = 2x \sin(1/x^2) - \frac{2}{x} \cos(1/x^2), x \neq 0$. This is unbounded and so not integrable (so $S(f, D)$ is not well-defined).

2012

1,3E $f(x) = e^{-x}\cos(1/x)$ then f does not attain upper bound or lower bound. For the second part, fix $\epsilon > 0$, there exists $K > 0$ such that $f(x) < \epsilon$ for all $|x| > K$. Then $f(x) \le \max\{\epsilon, M\}$ where $M = \max\{f(x) : |x| \le M\}$ (as f obtains maximum on closed interval $[-M, M]$) and so f is bounded. Pick ϵ small enough (so if we vary ϵ, we will vary K so as ϵ decreases, K will increase and M will increase) such that $\epsilon < M$ then we attain the upper bound at some x with $f(x) = M$.

1,4F $M\int_0^1 g(x)dx - \int_0^1 f(x)g(x)dx = \int_0^1 g(x)(M - f(x))dx$ and since $M - f(x) \ge 0$ for all $x \in [0,1]$ and $g(x) \ge 0$ so the integral is non-negative (if $h \ge 0$, then $\int_0^1 h(x)dx \ge 0$ because $s(D,h) \ge 0$ for all partition D).

Let $M = \max_{[0,1]} f$ and $m = \min_{[0,1]} f$, and if $I = \int_0^1 g(x)dx$,

$$m \le \frac{1}{I}\int_0^1 f(x)g(x)dx \le M.$$

As $f(x)$ is continuous so by IVT there exists α such that $f(\alpha) = \frac{1}{I}\int_0^1 f(x)g(x)dx$.

1,9E For all $\epsilon > 0$, there exists k_N (for each k), such that for all $n > k_N$, $|y_n^{(k)} - l| < \epsilon$. Thus, let $N = \max_k\{k_N\}$ and so for all $n > N$, $n > k_N$ and x_n is one of $y_n^{(k)}$ and so $|x_n - l| < \epsilon$.

Let $y_n^{(n)} = (-1)^n$ and $y_n^{(j)} = 1$ for all $n \ne j$. Then it is clear that for all j, the sequence $y_n^{(j)} \to 1$. But $x_n = (-1)^n$ and it diverges.

It is clear by AM-GM inequality that $b_{n+1} \ge a_{n+1}$ with equality if and only if $a_n = b_n$. Now $a_1 < b_1$ and so inductively $a_n < b_n$. Also

$$\frac{a_{n+1}}{a_n} = \frac{\sqrt{b_n}}{\sqrt{a_n}} > 1, b_{n+1} - b_n = \frac{a_n - b_n}{2} < 0$$

and so a_n is increasing and b_n is decreasing. As $a_n < b_n < b$ for all n so a_n is bounded above and so converges to some limit l_1. Similarly, b_n is bounded below by a and so it converges to l_2.

Finally, let $c_n = b_n - a_n = (\sqrt{b_n} - \sqrt{a_n})(\sqrt{b_n} + \sqrt{a_n})$ then $c_{n+1} = \frac{(\sqrt{b_n}-\sqrt{a_n})^2}{2}$ and so

$$\frac{c_{n+1}}{c_n} = \frac{\sqrt{b_n} - \sqrt{a_n}}{2(\sqrt{b_n} + \sqrt{a_n})} < \frac{1}{2}$$

and therefore $c_n \to 0$. So $l_1 = l_2$.

1,10D (i) We may assume $a_{n+1} - a_n \to 0$ because if we set $b_n = a_n - nl$ and so $b_{n+1} - b_n \to 0$ and $\frac{b_n}{n} = \frac{a_n}{n} - l$ so $\frac{b_n}{n}$ converges if and only if $\frac{a_n}{n}$ converges. So now for each $\epsilon > 0$, there exists N_1 such that for all $n > N_1$,

$$b_{n+1} - b_n < \frac{\epsilon}{2}$$

and so

$$b_{n+1} < b_n + \frac{\epsilon}{2} < b_{n-1} + \epsilon < \cdots < b_{N_1} + \frac{(n - N_1)\epsilon}{2}$$

and so

$$\frac{b_{n+1}}{n+1} < \frac{(n - N_1)\epsilon}{2(n+1)} + \frac{b_{N_1}}{n+1} < \frac{\epsilon}{2} + \frac{b_{N_1}}{n+1}$$

Now pick N large enough with $\frac{b_{N_1}}{N+1} < \frac{\epsilon}{2}$ so now for all $n > N$ we have $\frac{b_{n+1}}{n+1} < \epsilon$.

(ii) $a_n = -1$ if n is odd and $a_n = 1$ if n is even. Then clearly $\frac{a_n}{n} \to 0$ but $a_{n+1} - a_n$ does not converge because $a_{n+1} - a_n = 2$ if n is odd and $a_{n+1} - a_n = -2$ if n is even.

(iii) Let $a_n = \sum_{i=1}^{n} \frac{1}{i}$ then for each fixed k,

$$a_{n+k} - a_n = \sum_{i=n+1}^{n+k} \frac{1}{i} < \frac{k}{n} \to 0, \text{ as } n \to \infty.$$

(iv) Suppose a_n does not converge then as a_n is real so (a_n) is not Cauchy. There exists $\epsilon > 0$, for all n, there exists $m = g(n) > n$ such that $|a_m - a_n| > \epsilon$. Now take this function $f(n) = g(n) - n$ then $n + f(n) = g(n)$ and so $|a_{n+f(n)} - a_n| > \epsilon$ for all n which contradicts the assumption. So a_n converges.

1,11I (i) Fix $0 < x < 1$ and let $f^n(x) = a_n$ and $a_{n+1} = f(a_n) < a_n$ so a_n is decreasing and bounded below so a_n converges to a limit l. Suppose $l > 0$, then $a_{n+1} = f(a_n) \to f(l)$ and so we have $f(l) = l$. But for all $l > 0$ we must have $f(l) < l$ and therefore $l = 0$.

(ii) No. For example, let $f(x) = x^2$. Take $\epsilon = \frac{1}{2}$, such that for all n, we pick x big enough (close to 1) so that $x^{2^n} > \frac{1}{2}$ (as n is fixed so pick x close to 1).

(iii) Construct a sequence a_n with $a_1 = \frac{1}{2}$, $0 < a_{n+1} < a_n$ for all n and $a_n \to \frac{1}{4}$ as $n \to \infty$, and $a_n \neq \frac{1}{4}$ for all n. Now define the function f by $f(x) = \frac{x}{1000}$ if $x \neq a_n$ for any n and $f(a_n) = a_{n+1}$ for all n. Then $f(x) < x$ for $x \in (0,1)$ and pick $x = a_1$, $f^n(x) = a_n \to \frac{1}{4} \neq 0$.

(iv) No. Let $x = 0.a_1a_2\ldots$ be the decimal expansion and for convention if $x = 0.a_1\ldots a_n$ then we write $x = 0.a_1\ldots(a_n - 1)999999\ldots$ so that in this way the expansion is infinite. For example, if $x = 0.2$ we will write $x = 0.19999\ldots$. Let $x = 0.00\ldots a_1a_2\ldots$. Define $f(x)$ to be the function which switches the second non-zero term in the expansion of x to zero, i.e. $f(x) = 0.00\ldots a_1 0 a_3\ldots$. Then for each x, $f^{(n)}(x) \to 0.00\ldots a_1 > 0$.

1,12F Part (a) is book work. Let $h(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{(2n)!}$ then $h(x^2) = f(x)$. $h(x)$ has $R = \infty$ which means for all $x \in \mathbb{R}$ converges and so $h(x^2)$ converges for all x. Similarly, if $h(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{(2n+1)!}$ then $h(x)$ has $R = \infty$ by ratio test then $g(x) = xh(x^2)$ and so it converges for all $x \in \mathbb{R}$.

As f, g have $R = \infty$ so they are differentiable over $\mathbb{R}$. Further,

$$f'(x) = \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n-1}}{(2n-1)!} = -g, g'(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = f.$$

Let $h(x) = f^2(x) + g^2(x)$ then $h'(x) = 2(f(x)f'(x) + g(x)g'(x)) = 0$. Therefore, $h(x)$ is constant and $h(0) = f(0) + g(0)$ where $f(0) = 1$ and $g(0) = 0$ so $h(x) = 1$.

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1,3D $\exp(x) = 1 + x + x^2/2 + \cdots \geq 1 + x$ for all $x \geq 0$. By expanding the product of $\prod_{j=1}^{n}(1 + a_j)$ the first inequality is clear, and the second inequality comes from the fact $1 + a_j \leq \exp(a_j)$.

Suppose $\prod_1^n(1 + a_j)$ converges then as $a_j \geq 0$ so the sum $\sum a_j$ converges by the first inequality and conversely if $\sum_j a_j$ converges so does $\exp(\sum_j a_j)$ so the product converges by the second inequality.

1,4F (a) is alternating test. For (b), use condensation and comparison tests, the series diverges.

1,9D (a) The first one converges for all x (as expected because it is $\exp(x)$). The second one only converges at $x = 0$ (as expected because $n!x^n$ is unbounded for any $x \neq 0$). For the third one, let $a_n = (n!)^2 x^{n^2}$ then

$$\frac{a_{n+1}}{a_n} = (n+1)^2 x^{2n+1}.$$

Suppose $|x| > 1$ then $\frac{a_{n+1}}{a_n} \to \infty$ and so a_n is unbounded, so the series cannot converge. Suppose $|x| < 1$, then we claim that for n large enough, we have $b_n = (n!)^2 x^{n^2 - n} < 1$. As

$$\frac{b_{n+1}}{b_n} = (n+1)^2 x^{2n} \to 0$$

and so there exists n such that $b_{n+1} < \frac{b_n}{2}$. Hence for n large enough, $b_n < 1$ and this implies that

$$a_n = (n!)^2 x^{n^2} < x^n$$

and so by comparison test we conclude that the series $\sum_n a_n$ converges absolutely for $|x| < 1$. Therefore, $R = 1$.

(b) Let $f(x) = (1 + x)^{\frac{1}{2}}$ and for $x \in (0, 1)$ $f(x)$ is infinitely differentiable and

$$f^{(n)}(x) = \frac{1}{2}\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)\cdots(\frac{1}{2} - n + 1)(1 + x)^{\frac{1}{2} - n}$$

and the result follows by applying Taylor's theorem with some remainder and we check the remainder tends to 0 and the series converges. We have

$$\frac{c_{n+1}}{c_n} = \frac{2n - 1}{2n + 2} \to 1$$

and so it converges for all $x < 1$.

Now if you pick Lagrange's remainder then the remainder has the form, $c_n f^{(n)}(\theta x)$ and $f^{(n)}(\theta x) < 1$ and so the modulus of the remainder is less than $|c_n|$. But as the series converges so $c_n \to 0$.

1,10E (a) is book work. For (b)

(i) Clearly if f is strictly increasing then f is injective. Suppose $x \geq y$ then $f(x) \geq f(y)$ because f is strictly increasing and so if $f(x) < f(y)$ then $x < y$.

(ii) For $a < x < b$ we have $c < f(x) < d$ and so c is the minimum and d is the maximum. Suppose f is not increasing, so we have $a < x_1 < x_2 < b$ but $f(x_1) > f(x_2)$. Let M be the maximum of f on $[a, x_2]$ and as f is continuous take x_3 with $f(x_3) = M > f(x_2), M > c$ and clearly $x_3 < x_2$.

Therefore, there exists $x_4 \in (a, x_3)$ such that $f(x_4) = f(x_2)$ (if this is not clear, apply IVT to the function $g(x) = f(x) - f(x_2)$). As f is injective this gives a contradiction. This shows f is increasing. To show it is strictly increasing, use the fact f is injective.

(iii) f attains maximum at b and minimum at a. Suppose f is not continuous at b, then there exists a sequence $x_n \in [a, b]$ with $x_n \to b$ but $f(x_n) \not\to f(b)$. We can assume x_n is an increasing sequence (because it is bounded above and converges so we can remove all 'bad' points). Also, $f(x_n)$ is increasing as both f and x_n are, and it is bounded above so $f(x_n) \to l$. As $f(x_n) \not\to f(b)$ so $l < f(b)$ and pick r with $l < r < f(b)$ so by the intermediate property we have y such that $f(y) = r$. But then as $f(x_n) \leq l$ so $y > x_n$ for all n and so the only possibility is $y = b$ but $f(y) = r < f(b)$ which gives a contradiction. A similar argument shows f is continuous at a.

For general point $x \in (a, b)$, suppose f is not continuous at x. For any sequence $x_n \to x$ with x_n increasing and $x_n < x$, let $f(x_n) \to l_1$ (limit exists as it is bounded above) and $y_n \to x$ with y_n decreasing and $y_n > x$, let $f(y_n) \to l_2$. As f is not continuous at x, we must have either some sequence x_n described above with $l_1 \neq f(x)$ or some sequence y_n with $l_2 \neq f(x)$. We may assume $l_1 \neq f(x)$. Then pick r with $l_1 < r < f(x)$ and so we have some y with $f(y) = r$. As $f(y) > l_1$ we have $y > x_n$ for all n but $x_n \to x$ so $y = x$, which contradicts $f(y) < f(x)$.

1,11E The first two parts are book work. For the third part, consider the function

$$f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$$

and apply Rolle's theorem so we get the Cauchy mean value theorem. Then

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(z)}{g'(z)}, z \in (a, x).$$

For all $\epsilon > 0$, there exists $\delta > 0$ such that for all $z - a < \delta$,

$$l - \epsilon < \frac{f'(z)}{g'(z)} < l + \epsilon,$$

and so for all $x - a < \delta$,

$$l - \epsilon < \frac{f(x)}{g(x)} = \frac{f'(z)}{g'(z)} < l + \epsilon, z < x < a + \delta.$$

For the last part, apply (iii) once we have

$$\lim_{h \to 0} \frac{f(a+h) - f(a-h) - 2f(a)}{h^2} = \lim_{h \to 0} \frac{f'(a+h) - f'(a-h)}{2h}$$

(where the numerator tends to 0 as $h \to 0$ because f is continuous). Then since f is twice differentiable, we have

$$\lim_{h \to 0} \frac{f'(a+h) - f'(a-h)}{2h} = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{2h} + \lim_{h \to 0} \frac{f'(a) - f'(a-h)}{2h} = \frac{f''(a)}{2} + \frac{f''(a)}{2} = f''(a).$$

1,12F The first two are book work. Then for any D_1, D_2 we have

$$s(f, D_1) \leq s(f, D_1 \cup D_2) \leq S(f, D_1 \cup D_2) \leq S(f, D_2).$$

We have

$$s(f, D) \leq p(f, D) \leq \exp(s(f, D))$$

(if you are not clear see question [1,3D]). Since f is integrable, $\int_a^b f(x)dx$ exists and $s(f, D) \leq \int_a^b f(x)dx$. Then $p(f, D) \leq \exp(\int_a^b f(x)dx)$ because exp is increasing.