Linear Algebra 1

zc231

1. (a),(c),(e),(h). Note that (h) is the trivial space. For the rest of them it suffices to find some two functions in the space such that the sum of them is not in the space. For example, for (b), $f_1(t) = f_2(t) = 4/5$ but $(f_1 + f_2)(t) = 8/5 > 1$.

2. By considering the change of basis matrix, we conclude that (a) is a basis. For (b), if n is even, then
$$(e_1 + e_2) - (e_2 + e_3) + \cdots + (e_{n-1} + e_n) - (e_n + e_1) = 0$$
and so the vectors are linearly dependent. If n is odd, let $f_i = e_i + e_{i+1}$ where by convention we let $e_{n+1} = e_1$. Suppose $\sum_i a_i f_i = 0$ then we have $\sum_i (a_i + a_{i+1}) e_i = 0$ where by convention we let $a_{n+1} = a_1$. So we must have $a_i + a_{i+1} = 0$ for all i and so
$$a_1 = -a_2 = a_3 = \cdots = a_n = -a_1.$$
This shows that $a_1 = 0$ and so $a_i = 0$ for all i. So the vectors form a basis.

   For (c), we check whether these vectors are linearly independent. Let $f_1 = e_1 - e_n$, $f_2 = e_2 + e_{n-1}$ etc. so in general $f_i = e_i + (-1)^i e_{n-i+1}$. Suppose $\sum_i c_i f_i = 0$. Then we have
$$\sum_i c_i e_i + c_i(-1)^i e_{n-i+1} = 0.$$
We collect the coefficient of e_i for each i and so $\sum_i c_i f_i = \sum_i d_i e_i$ where
$$d_i = c_i + (-1)^{n+1-i} c_{n+1-i}.$$
But $e_1, \ldots, e_n$ are linearly independent and so $\sum_i d_i e_i = 0$ implies $d_i = 0$ for all i. Therefore $c_i = (-1)^{n-1} c_{n+1-i}$ for all i. Replace i by $n + 1 - i$ so we also have
$$c_{n+1-i} = (-1)^{i-1} c_i.$$
Therefore,
$$c_i = (-1)^{n-1} c_{n+1-1} = (-1)^{n-i}(-1)^{i-1} c_i = (-1)^{n-1} c_i.$$
This holds for all i and so either $c_i = 0$ for all i, or $n - 1$ is even. Therefore, when n is even, these vectors are linearly independent and when n is odd these are linearly dependent, in which case they do not form a basis.

3. (i) Suppose not, which means there exists $x \in T \backslash U$ and $y \in U \backslash T$. We claim that $x + y \notin T \cup U$ because if $x + y = z \in T$ then $y = z - x = z + (-x) \in T$ which is a contradiction.

   (ii) Consider the lines $T = \langle (1,0) \rangle$, $U = \langle (0,1) \rangle$ and $W = \langle (1,1) \rangle$ in $\mathbb{R}^2$. Then (a) $U \cap W = (0,0)$ and so $T + (U \cap W) = T$. But $T + U = T + W = \mathbb{R}^2$. For (b) we use the same example. We have $(T + U) \cap W = W$. But $(T \cap W) + (U \cap W) = (0,0)$.

   (iii) (a) For each $x \in T + (U \cap W)$, write $x = y + z$ where $y \in T$ and $z \in U \cap W$. Then $y + z \in T + U$ and $y + z \in T + W$. So
$$T + (U \cap W) = (T + U) \cap (T + W).$$
(b) For each $x \in (T \cap W) + (U \cap W)$, write $x = y + z$ where $y = T \cap W$ and $z \in U \cap W$. Then $y + z \in T + U$ and $y + z \in W$. So $x \in W$ and so
$$(T \cap W) + (U \cap W) \subset (T + U) \cap W.$$

4. (a) $V \to W$ by $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, x_4, -x_1 - x_2 - x_3 - x_4)$. (b) No such isomorphism exists because W has larger dimension. (c) For each function $f(x)$ in V, define $g(x)$ such that for each $x \in [-1, 1]$, $g(x) = f\left(\frac{x+1}{2}\right) \in W$. The inverse is $f(x) = g(2(x - 1/2))$. (d) For each $f(x) \in V$, define $g(x) = \int_0^x f(t)dt$. Then $g(x) \in W$ by fundamental theorem of calculus. (e) The functions in W have the form $w_{\alpha,\beta} = \alpha \cos t + \beta \sin t$ and so we have an isomorphism $V \to W$ by $(\alpha, \beta) \mapsto w_{\alpha,\beta}$. (f) W has infinite dimension but V has finite dimension. For example, the subspaces $W_n = \{p \in P : \deg p \leq n\}$ is a subspace of W_n for all n. (g) V has a countable basis $\{1, x, x^2, \ldots\}$, but W has an uncountable basis. Consider the subspace W' of W which consists of sequences $(\alpha_n)_{n=1}^{\infty}$ where $\alpha_n = n^\alpha$. It is clear that these sequences are linearly independent. To convince yourself, suppose they are not, then you have a finite linear combinations $\sum_i a_i v_i = 0$ where the nth term of each vector v_i is n^{α_i}. If some $a_i \neq 0$ then take j to be the one such that $a_j \neq 0$ and α_j is the largest and WLOG we assume $\alpha_j > 0$. Then as $n \to \infty$, $a_j n^{\alpha_j}$ dominates the other terms and so the sum of these cannot be zero.

5. (i) For both cases we can take α to be the identity and $\beta = -\alpha$. In general, for (a), for each $y = (\alpha + \beta)(x) \in Im(\alpha + \beta)$, we have $y = \alpha(x) + \beta(x) \in Im(\alpha) + Im(\beta)$ and so $Im(\alpha + \beta) \subset Im(\alpha) + Im(\beta)$. For (b), take $x \in \ker(\alpha) \cap \ker(\beta)$ and so $\alpha(x) = \beta(x) = 0$. Then $(\alpha + \beta)(x) = 0$ and so $\ker(\alpha) \cap \ker(\beta) \subset \ker(\alpha + \beta)$.

   (ii) For each $x \in V$, write $x = \alpha x + (x - \alpha x)$. Since
   $$\alpha(x - \alpha x) = \alpha x - \alpha^2 x = \alpha x - \alpha x = 0$$
   we conclude that $x \in Im(\alpha) + \ker(\alpha)$. To see this is a direct sum, take $y \in Im(\alpha) \cap \ker(\alpha)$. Then $y = \alpha x$ for some $x \in V$. But $y \in \ker(\alpha)$ and so $\alpha y = 0$. But
   $$\alpha y = \alpha^2 x = \alpha x = y$$
   and so $y = 0$.

6. We have
   $$\mathbb{R}^3 \cong U,\ (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, -x_1 - x_3, -2x_1 - 2x_2)$$
   and
   $$\mathbb{R}^2 \cong W,\ (x_1, x_2) \mapsto (x_1, x_2, x_2, x_2, -x_1).$$
   Therefore we conclude that
   $$\mathbb{R} \cong U \cap W,\ x \mapsto (-2x, x, x, x, 2x).$$
   So $e_1 = (-2, 1, 1, 1, 2)$ is the basis for $U \cap W$. Define $e_2 = (1, 0, 0, -1, 0), e_3 = (0, 1, 0, 0, -2)$ and $e_4 = (0, 1, 1, 1, 0)$. Then e_1, e_2, e_3 form a basis for U and e_1, e_4 form a basis for W. Also e_1, e_2, e_3, e_4 form a basis for $U + W$. Finally, let
   $$V = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 + 2x_2 + x_5 = x_3 + x_4\}$$
   then V has dimension 4. But V contains both U and W, so by considering the dimensions we conclude that $V = U + W$.

7. It is obvious that $r(\alpha) \geq r(\alpha|_W)$. By Rank-Nullity we have $\dim U = r(\alpha) + \dim(\ker \alpha)$. Since $\dim(\ker \alpha) \geq \dim(\ker \alpha|_W)$, so
   $$r(\alpha|_W) + \dim(\ker \alpha) \geq \dim W.$$
   Finally write $\dim(\ker \alpha) = \dim U - r(\alpha)$.

   For the first equality to hold we can just take α to be the zero map. For the second one we take α to be the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ with respect to the standard basis in $\mathbb{R}^2$. Let $W = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$. Then $W = \ker \alpha$. So $r(\alpha|_W) = 0$, $r(\alpha) = 1, \dim W = 1$ and $\dim U = 2$. So the second equality holds.

8. (i) Since $\alpha^{n+1}(x) = \alpha^n(\alpha x)$ and so $Im(\alpha^n) \supset Im(\alpha^{n+1})$. So $r_k \geq r_{k+1}$. Also if $\alpha^n(x) = 0$ then $\alpha^{n+1}(x) = \alpha(0) = 0$. So $\ker(\alpha^n) \subset \ker(\alpha^{n+1})$.

   For each k, by $r_k = Im(\alpha^k)$. Consider $\alpha_k : Im(\alpha^k) \to V$ where α_k is the restriction of α on $Im(\alpha^k)$. Then by Rank-Nullity,

$$r_k - r_{k+1} = \dim(\ker \alpha_k)$$

   and similarly, $r_{k+1} - r_{k+2} = \dim(\ker \alpha_{k+1})$. But $\ker \alpha_{k+1} \subset \ker \alpha_k$, so $r_k - r_{k+1} \geq r_{k+1} - r_{k+2}$. If $r_k = r_{k+1}$ then the above inequality gives $0 \geq r_{k+1} - r_{k+2}$. But $r_{k+1} - r_{k+2} \geq 0$ and so $r_{k+1} = r_{k+2}$. Similarly $r_{k+2} = r_{k+3}$ etc. and so $r_{k+l} = r_k$ for all l.

   (ii) Since $\alpha^2 \neq 0, \alpha^3 = 0$ we have $r_3 = 0$ and $r_2 > 0$. By (i) we have

$$r_1 - r_2 \geq r_2 - r_3 = r_2, \ r_0 - r_1 \geq r_1 - r_2$$

   and so $r_1 \geq 2r_2$. Since $r_0 = 5$ we have $5 = r_0 \geq 2r_1 - r_2 \geq 3r_2$. This shows that $r_2 = 1$ and $r_1 \geq 2$. Using $r_2 = 1$ we have $5 - r_1 \geq r_1 - 1$ and so $r_1 \leq 3$. So $r_1 = 2, 3$.

9. We have

$$\alpha e_1 = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}, \alpha e_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \alpha e_3 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

   and

$$\alpha e_1 = 2e_1 + e_2, \ \alpha e_2 = 2e_2 + e_3, \ \alpha e_3 = 2e_3.$$

   So the matrix is

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

   Take standard basis for the domain and

$$\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

   for the basis of the range.

10. (i) implies (ii) is obvious. (ii) implies (iii): It is clear that the B_i are pairwise disjoint because $U_j \cap \sum_{i \neq j} U_i = \{0\}$. The union spans U because each B_i is a basis for U_i. So we need to check that they are linearly independent. Let $B_i = \{b_{i,1}, \ldots, b_{i,i_k}\}$ for each i. Suppose we have c_{ij} such that

$$\sum_i \sum_{j=1}^{i_k} c_{ij} b_{i,j} = 0.$$

   Then

$$\sum_{j=1}^{1_k} c_{1j} b_{1,j} = -\sum_{i>1} \sum_{j=1}^{i_k} c_{ij} b_{i,j}.$$

   So

$$\sum_{j=1}^{1_k} c_{1j} b_{1,j} \in U_1 \cap \sum_{i \neq 1} U_i = \{0\}$$

   and so $\sum_{j=1}^{1_k} c_{1j} b_{1,j} \in U_1 = 0$. Since B_1 is a linear independent set, so $c_{1j} = 0$ for all j. Then we start with

$$\sum_{i>1} \sum_{j=1}^{i_k} c_{ij} b_{i,j} = 0$$

and apply the above repeatedly. So we conclude that $c_{ij} = 0$ for all i, j.

(iii) implies (i): Since the union of the B_i is a basis for $U = \sum_i u_i$. Then for each $u \in U$, u can be written as $\sum_i u_i$ for some $u_i \in U$. For uniqueness, it suffices to show that if $\sum_i u_i = 0$ then $u_i = 0$ for each i. Write u_i as a linear combination of vectors in B_i for each i. So

$$\sum_i u_i = \sum_i \sum_j c_{ij} b_{i,j} = 0.$$

But the B_i are disjoint and their union is a basis so $c_{ij} = 0$ for all i. This shows that $u_i = 0$ for each i.

For the last part, consider the lines $U_1 = \langle(1,0)\rangle, U_2 = \langle(0,1)\rangle$ and $U_3 = \langle(1,1)\rangle$ in $\mathbb{R}^2$.

11. The zero map is clearly in R. If $\alpha, \beta \in R$ then for each $y \in Y$, we have

$$(\alpha + \beta)(y) = \alpha(y) + \beta(y) \in Z + Z = Z$$

and for each c we have $c\alpha(y) \in \alpha(cy) \in Z$. So this is a subspace.

Pick a basis $\{v_1, \dots, v_n\}$ for V such that $\{v_1, \dots, v_y\}$ is a basis for Y for some $y \leq n$ and a basis $\{w_1, \dots, w_m\}$ for W such that $\{w_1, \dots, w_z\}$ is a basis for Z for some $z \leq m$. So $\dim Y = y$ and $\dim Z = z$. Then for each $\alpha \in R$, for each $1 \leq i \leq y$ we must have $\alpha(v_i) = w_j$ for some $1 \leq j \leq z$ and for $i > y$ we have $\alpha(v_i) = w_j$ for any j. Therefore, the dimension is

$$yz + (n-y)m = nm - ym + yz = \dim V \dim W - \dim Y \dim W + \dim Y \dim Z.$$

12. Let $S = \{1, \dots, n\}$ and i_1 be the least number in S such that $e_{i_1} \notin U$. Let $U_1 = U + e_{i_1}$. For each $j \geq 1$, let i_j be the least number in S such that $e_{i_j} \notin U_j$ and define $U_{j+1} = U_j + e_{i_j}$. Repeat this until $U_k = \mathbb{F}^n$ for some k. Let $I = \{i_1, \dots, i_k\}$ and $W = \langle e_{i_1}, \dots, e_{i_k}\rangle$. Then clearly $U + W = U_k = \mathbb{F}^n$. We need to check that $U \cap W = \{0\}$.

Suppose $x \in U \cap W$ then there exist a_j such that $x = \sum_j a_j e_{i_j}$ because $x \in W$. Since $x \in U \subset U_{k-1}$ so we have

$$x = \sum_{j \in I, j \neq k} a_j e_{i_j} + a_{i_k} e_{i_k} \in U_{k-1}.$$

But $\sum_{j \in I, j \neq i_k} a_j e_j \in U_{k-1}$ and so $a_{i_k} e_{i_k} \in U_{k-1}$. But by construction $e_{i_k} \notin U_k$ and so $a_{i_k} = 0$. Now repeat this for U_{k-2} etc. and so $a_j = 0$ for all $j \in I$. Therefore $x = 0$.

13. Consider $V = \mathbb{R}^3$ and

$$e_1 = \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix} \right\rangle, e_2 = \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle, e_3 = \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle, e_4 = \left\langle \begin{pmatrix}0\\1\\1\end{pmatrix} \right\rangle.$$

Let $X = \{e_1, e_2\}$ and $Y = \{e_3, e_4\}$. Then X, Y satisfy the condition. But $X \cup Y$ is linearly dependent.

14. We prove the following statement: If V, W are proper subspaces of U then there exists $u \in U$ such that $u \notin V$ and $u \notin W$. Take $v, w \in U$ such that $v \notin V$ and $w \notin W$. If $v \notin W$, take $u = v$ and if $w \notin V$, take $u = w$. So we now assume $v \in W$ and $w \in V$. Take $u = v + w$. If $u \in V$ then $v = u - w = u + (-w) \in V$ which is a contradiction. So $u \notin V$ and similarly $u \notin W$. This completes the proof.

Let $U = \mathbb{R}^n$ and V, W be subspaces of U. Assume $\dim V = \dim W = m \leq n$ and let $d = n - m$. If $d = 0$ then $V = W = U$ so there is nothing to prove. If $d > 0$, define $V_0 = V$ and $W_0 = W$.

For each j with $0 < j \leq d$, pick a vector u_j such that $u_j \notin V_{j-1}$ and $u_j \notin W_{j-1}$ and then define $V_j = V_{j-1} + u_j$ and $W_j = W_{j-1} + w_j$. Let $X = \langle u_1, \dots, u_d \rangle$. Then

$$V + X = V_d = U, \ W + X = W_d = U.$$

If $x \in V \cap X$, then $x \in X$ and so we can write $x = \sum_j a_j u_j$ for some a_j. But $x \in V \subset V_{d-1}$ and so

$$x = (\sum_{j \neq d} a_j u_j) + a_d u_d \in V_{d-1}$$

which implies that $a_d u_d \in V_{d-1}$. So $a_d = 0$. Repeat this for $V_{d-2}, \dots$ etc. so we conclude that $a_j = 0$ for all j and so $x = 0$. So $V \cap X = \{0\}$. Similarly $W \cap X = \{0\}$.

15. Let θ_1, θ_2 be linear maps in $L(U, V)$ and $c \in \mathbb{F}$. Then

$$\Phi(\theta_1+\theta_2)(x) = \beta(\theta_1+\theta_2)\alpha(x) = \beta(\theta_1\alpha(x)+\theta_2\alpha(x)) = \beta\theta_1\alpha(x)+\beta\theta_2\alpha(x) = \Phi(\theta_1)(x)+\Phi(\theta_2)(x)$$

for all x because β is linear. So $\Phi(\theta_1 + \theta_2) = \Phi(\theta_1) + \Phi(\theta_2)$. Similarly,

$$\Phi(c\theta)(x) = \beta(c\theta)\alpha(x) = c\beta\theta\alpha(x) = c\Phi(\theta)(x).$$

Let k the dimension of the kernel of Φ and so the rank of Φ is $\dim U \dim V - k$. $\theta \in \ker \Phi$ if and only if $\beta\theta\alpha(x) = 0$ for all $x \in T$. This means $\theta\alpha(x) \in \ker B$ for all $x \in T$ and so we can rewrite this as $\theta(\alpha T) \subset \ker B$. Since $\dim(\alpha T) = r$ and $\dim \ker B = \dim V - s$, so by Question 11 we have

$$k = \dim(\ker \Phi) = \dim U \dim V - r \dim V + r(\dim V - s) = \dim U \dim V - rs.$$

Therefore, the rank of Φ is

$$\dim U \dim V - k = rs.$$

Alternatively, we can pick a basis for the kernel of α first and then extend this to a basis for T and pick a basis for the range of β first and then extend this to a basis for W. Then the matrix we obtain for α and β will be $\alpha = \begin{pmatrix} A & 0 \end{pmatrix}$ and $B = \begin{pmatrix} B \\ 0 \end{pmatrix}$ where A has r columns and B has s rows. Then if you take $\beta \circ \theta \circ \alpha$ you would get a matrix which look like $\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}$ where C has r columns and s rows. So the rank of Φ is rs.