Linear Algebra 2

zc231

1. Row swapping matrix $S(kl)$: $S_{ij} = 1$ if $i = j \neq k, l$, $S_{kl} = S_{lk} = 1$ and $S_{ij} = 0$ otherwise. We have $S^{-1} = S$.

   Row multiplication matrix $M(m)$: $M_{ij} = 1$ if $i = j \neq k$, $M_{kk} = m$ and $M_{ij} = 0$ otherwise. The inverse is $M^{-1}(m) = M(1/m)$.

   Row addition matrix $A(m)$: $A_{ij} = 1$ if $i = j$, $A_{ik} = m$ for some $k \neq i$ and $A_{ij} = 0$ otherwise. The inverse is $A^{-1}(m) = A(-m)$.

   Each elementary matrix is invertible and so a product of elementary matrices is invertible. Conversely, let T be an invertible matrix. Use row swapping matrix if necessary so that the top left entry is non-zero. Then use the row multiplication matrix so that $T_{1,1} = 1$. Apply row addition matrix so that the first column is e_1.

   For the second column, use row swapping matrix if necessary so that the entry $T_{2,2}$ is non-zero. Note that there must exist $i \geq 2$ such that $T_{2,i} \neq 0$ because otherwise the second column is a multiple of the first column. Then repeat the above so that the second column is e_2. Repeat this for the rest so that we conclude we end up with the identity matrix. Since the inverses of elementary matrices are again elementary matrices, so taking inverse shows that T is a product of elementary matrices.

   Let A be the given matrix. We have

$$\begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} A = I$$

   Therefore,

$$A^{-1} = \begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 \\ 0 & 1 & 0 \end{pmatrix}$$

2. A can be written as BC where $B \in \mathrm{Mat}_{m,s}(\mathbb{F})$ and $C \in \mathrm{Mat}_{s,n}(\mathbb{F})$. Write A as $(v_1|v_2|\cdots|v_r|v_{r+1}|\cdots|v_n)$ such that $v_1, \ldots, v_r$ are linearly independent and for each $i > r$,

$$v_i = \sum_{j \leq r} a_{ij} v_j$$

   for some a_{ij}. Set $B = (v_1|v_2|\cdots|v_r)$ and let

$$C = (e_1|e_2|\cdots|e_r|A_i|\cdots|A_n)$$

   where $e_1 = (1, 0, 0, \ldots)^T$ etc. are the standard vectors and A_i are the vectors such that the jth entry of A_i is a_{ij}. Then $A = BC$.

   If $A = BC$ for $B \in \mathrm{Mat}_{m,s}(\mathbb{F})$ and $C \in \mathrm{Mat}_{s,n}(\mathbb{F})$, then the rank of A is at most s. So $s \geq r$. This shows that r is the least such integer.

   Let the column rank of A^T be s. Then s is the least integer for which $A^T = B'C'$ with $B' \in \mathrm{Mat}_{n,r}(\mathbb{F})$ and $C' \in \mathrm{Mat}_{r,m}(\mathbb{F})$. But $A^T = C^T B^T$ where $C^T \in \mathrm{Mat}_{n,r}(\mathbb{F})$ and $B^T \in \mathrm{Mat}_{r,m}(\mathbb{F})$. So $s = r$.

3. You can obtain these by setting $\xi_j' = \sum_j a_j \xi_j$ and apply $\xi_j'(x_i') = \delta_{ij}$. (a) $\xi_2, \xi_1, \xi_4, \xi_3$. (b) $\xi_1, \xi_2/2, 2\xi_3, \xi_4$. (c) $\xi_1, -\xi_1+\xi_2, \xi_1-\xi_2+\xi_3, \xi_4-\xi_3+\xi_2-\xi_1$. (d) $\xi_1+\xi_2, \xi_2+\xi_3, \xi_3+\xi_4, \xi_4$. (d) can be obtained by the answer of (c) and taking dual twice.

4. Since P_n has dimension $n+1$ so P_n^* has dimension $n+1$. It suffices to check $\epsilon_0, \dots, \epsilon_n$ are linearly independent. Suppose we have $a_0, \dots, a_n$ such that $\sum_{j=0} a_j \epsilon_j = 0$. Then $\sum_j a_j \epsilon_j(x^i) = 0$ for all i. But $\epsilon_j(x^i) = j^i$ and so we have $\sum_j a_j j^i = 0$ for all i. This can be written as $Av = 0$ where $v = (a_0, a_1, \dots, a_n)^T$ and

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 0 \\ 0 & 1 & 2 & \dots & n \\ 0 & 1^2 & 2^2 & \dots & n^2 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 1^n & 2^n & \dots & n^n \end{pmatrix}.$$

But $\det A \neq 0$ (Vandermonde matrix) so $a_0 = a_1 = \cdots = a_n = 0$. So these are linearly independent.

The kth polynomial in the basis of P_n dual to $\epsilon_0, \dots, e_n$ should be $p_k(x)$ such that $p_k(k) = 1$ and $p_k(i) = 0$ for all $i \neq k$. So if

$$f_k(x) = x(x-1)\cdots(x-(k-1))(x-(k+1))\cdots(x-n)$$

then $p_k(x) = \frac{f_k(x)}{f_k(k)}$.

5. (a) It suffices to show that if $x \neq 0$ then there is a linear map θ such that $\theta(x) \neq 0$. Pick a basis $\{e_1, \dots, e_n\}$ for V and a dual basis $\{\theta_1, \dots, \theta_n\}$ for V^*. Let $x = \sum_i a_i e_i$. Suppose $\theta(x) = 0$ for all $\theta \in V^*$. Then $\theta_j(x) = 0$ for each j and so $a_j = 0$ for each j. So $x = 0$.

   (b) If $A \subset B$, then for each $\theta \in B^\circ$, $\theta(v) = 0$ for all $v \in B$ and so $\theta(v) = 0$ for all $v \in A$. So $\theta \in A^\circ$ and so $B^\circ \subset A^\circ$. The converse can be proved by taking dual twice, so that $A^{\circ\circ} = A$ and $B^{\circ\circ} = B$.

   $A = V$ if and only if $V \subset A$, if and only if $A^\circ \subset V^\circ = \{0\}$.

6. We have

$$\tau_A(B_1 + B_2) = \mathrm{Tr}(A(B_1+B_2)) = \mathrm{Tr}(AB_1) + \mathrm{Tr}(AB_2) = \tau_A(B_1) + \tau_A(B_2)$$

   and

$$\tau_A(\lambda B) = \mathrm{Tr}(\lambda AB) = \lambda \,\mathrm{Tr}(AB) = \lambda \tau_A(B)$$

   so the map is linear for each A.

   Let $f(A) = \tau_A$. f is linear because

$$f(A_1 + A_2)(B) = \mathrm{Tr}((A_1+A_2)B) = \mathrm{Tr}(A_1 B) + \mathrm{Tr}(A_2 B) = f(A_1)(B) + f(A_2)(B)$$

   and

$$f(\lambda A)(B) = \mathrm{Tr}(\lambda AB) = \lambda \,\mathrm{Tr}(AB) = \lambda f(A)(B)$$

   for all B. So $f(A_1 + A_2) = f(A_1) + f(A_2)$ and $f(\lambda A) = \lambda f(A)$.

   It is injective because if A is in the kernel of f, then $f(A)(B) = 0$ for all B. Then $\mathrm{Tr}(AB) = 0$ for all B. Take B with $B_{kl} = 1$ for some fixed k, l and $B_{ij} = 0$ for all other entries. Then $\mathrm{Tr}(AB) = 0$ implies $A_{kl} = 0$. Since k, l are arbitrary, we conclude that $A_{kl} = 0$ for all k, l and so $A = 0$. Finally, by comparing the dimensions and Rank-Nullity, we conclude that f is an isomorphism.

7. (i) We have

$$\mathrm{Tr}(AB) = \sum_{i,j} A_{ij}B_{ji} = \sum_{i,j} B_{ij}A_{ji} = \mathrm{Tr}(BA)$$

and so if $\alpha\beta - \beta\alpha = I$, then

$$0 = \mathrm{Tr}(AB - BA) = \mathrm{Tr}(I) = \dim(V) > 0$$

which is a contradiction.

(ii) Let $\alpha = \frac{d}{dx}$ and β be the multiplication by x. Then for any function $f(x)$, we have

$$\frac{d}{dx}(xf(x)) - x\frac{d}{dx}(f(x)) = x\frac{d}{dx}(f(x)) + f(x) - x\frac{d}{dx}(f(x)) = f(x)$$

and so $\alpha\beta - \beta\alpha$ is the identity.

8. It suffices to show that row (column) swapping, row (column) multiplication and row (column) addition on a matrix given by bilinear form corresponds to a change of basis. Indeed, let $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ be bases for U and V respectively. So $A_{ij} = \phi(e_i, f_j)$. Then swapping the ith and the jth row (column) corresponds to swapping e_i and e_j (f_i and f_j). Multiplying the ith row (column) by m corresponds to replacing e_i by me_i (f_i by mf_i). Finally, add m times the jth row (column) to the ith row (column) corresponds to replacing e_i by $e_i + me_j$ (f_i by $f_i + mf_j$). Each of these is invertible so a sequence of these actions correspond to a change of basis and eventually we can transform the given matrix to the one such that the top left $r \times r$ square matrix is identity and the rest of the entries are zero because the rank is r.

   The dimension of the left kernel is $\dim U - r$ and the dimension of the right kernel is $\dim V - r$.

9. The idea is to add A times the top half of the matrix. For any matrices A, B the ith row of AB is obtained by the ith row of A multiplied by B on the right. So add the ith row of the bottom half by $a_{i1}V_1 + a_{i2}V_2 + \cdots + a_{in}V_n$ where V_i is the ith row of the top half. This gives us the matrix D.

   Since $\det C = \det A \det B$ and $\det D = \det(AB)$ so we conclude that $\det A \det B = \det(AB)$.

10. (i) Let $C = \mathrm{adj}(AB)$. Then $ABC = \det(AB)I$. But

$$AB\,\mathrm{adj}(B)\,\mathrm{adj}(A) = \det A \det BI = \det(AB)I$$

so $\mathrm{adj}(AB) = \mathrm{adj}(B)\,\mathrm{adj}(A)$.

(ii) We have

$$\det A \det(\mathrm{adj}(A)) = \det(A\,\mathrm{adj}(A)) = \det(\det AI) = (\det A)^n.$$

So $\det(\mathrm{adj}(A)) = (\det A)^{n-1}$.

(iii) Let $C = \mathrm{adj}(\mathrm{adj}(A))$ then C is the unique matrix such that $C\,\mathrm{adj}(A) = \det(\mathrm{adj}(A))I$. By (ii) we conclude that $C\,\mathrm{adj}(A) = (\det A)^{n-1}I$. On the other hand,

$$(\det A)^{n-2}A\,\mathrm{adj}(A) = (\det A)^{n-1}I$$

and so $C = (\det A)^{n-2}A$.

If A is singular, then there exists an open ball $B(0, r)$ such that for all $\lambda \in B(0, r)$, $A + \lambda I$ is invertible. (i) We have

$$\mathrm{adj}(AB + \lambda B) = \mathrm{adj}(B)\,\mathrm{adj}(A + \lambda I).$$

Each entry of $\operatorname{adj}(AB + \lambda B)$ is a polynomial in λ and so it is continuous as a function in λ. Therefore,

$$\operatorname{adj}(AB) = \lim_{\lambda \to 0} \operatorname{adj}(AB + \lambda B) = \operatorname{adj}(B) \lim_{\lambda \to 0} \operatorname{adj}(A + \lambda I) = \operatorname{adj}(B) \operatorname{adj}(A).$$

(ii) $\det(\operatorname{adj}(A + \lambda I))$ is again a polynomial in λ, so

$$\det(\operatorname{adj} A) = \lim_{\lambda \to 0} \det(\operatorname{adj}(A + \lambda I)) = \lim_{\lambda to 0} (\det(A + \lambda I))^{n-1} = \det A = 0.$$

(iii) Each entry of $\operatorname{adj}(\operatorname{adj} A + \lambda)$ is a polynomial in λ so again by continuity argument we conclude that $\operatorname{adj}(\operatorname{adj} A) = (\det A)^{n-2} A$.

Clearly if A is invertible, so is $\operatorname{adj}(A)$. If $r(A) \leq n - 2$ then every $n - 2 \times n - 2$ submatrix is singular and so every $n - 2$ minor is zero. Therefore $\operatorname{adj}(A) = 0$ and so the rank is 0. If $r(A) = n - 1$, let $B = \operatorname{adj}(A)$ and so $AB = \det AI = 0$. So the image of B is contained in the kernel of A. By Rank-Nullity, $\dim \ker(A) = 1$ and so $r(B) \leq 1$. But $r(A) = n - 1$ and so every $n - 1$ minor is non-zero. In particular, $r(B) \neq 0$ and so $r(B) = 1$.

11. The map is clearly linear. Suppose ξ is in the kernel. Then $\xi(t^i) = 0$ for all $i \geq 0$ and so $\xi = 0$. Therefore, the map is an isomorphism.

(a) $D(t^i) = it^{i-1}$ so

$$(a_0, a_1, a_2, a_3, \ldots) \mapsto (0, a_0, 2a_1, 3a_2, 4a_3, \ldots).$$

(b) $S(t^i) = t^{2i}$ so

$$(a_0, a_1, a_2, a_3, \ldots) \mapsto (a_0, a_2, a_4, a_6, \ldots).$$

(c) $E(t^i) = (t-1)^i = \sum_{j=0}^{i} \binom{i}{j} (-1)^j t^{i-j}$. So

$$(a_0, a_1, a_2, a_3, \ldots) \mapsto (a_0, a_1 - a_0, a_2 - 2a_1 + a_0, a_3 - 3a_2 + 3a_1 - a_0, \ldots).$$

(d) We have $DS(t^i) = D(t^{2i}) = 2it^{2i-1}$ so

$$(a_0, a_1, a_2, a_3, \ldots) \mapsto (0, 2a_1, 4a_3, 6a_5, \ldots).$$

(e) We have $SD(t^i) = it^{2i-2}$ so

$$(a_0, a_1, a_2, a_3, \ldots) \mapsto (0, a_0, 2a_2, 3a_4, 4a_6, \ldots).$$

If we do D^* and then S^*, then by (a) and (b) we have

$$(a_0, a_1, a_2, a_3, \ldots) \mapsto (0, a_0, 2a_1, 3a_2, 4a_3, \ldots) \mapsto (0, 2a_1, 4a_3, 6a_5, \ldots)$$

which is the same as what we have in (d).

If we do S^* and then D^*, by (a) and (b) we have

$$(a_0, a_1, a_2, a_3, \ldots) \mapsto (a_0, a_2, a_4, a_6, \ldots) \mapsto (0, a_0, 2a_2, 3a_4, 4a_6, \ldots)$$

which is the same as what we have in (e).

12. Let $x \in U \cap W$. Then $x \in W$ implies that $\phi(x, u) = 0$ for all $u \in U$. But $x \in U$ and $\phi|_{U \times U}$ is non-singular, so $x = 0$. Therefore $U \cap W = \{0\}$. Since $\dim U + \dim W = \dim W + \dim W^{\perp} \geq \dim V$ so $U \oplus W = V$. So the restriction on W is non-singular. Now pick bases for W and U respectively, and then the matrix of ϕ looks like

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

where A is the restriction $\phi|_{W \times W}$ and C is the restriction $\phi|_{U \times U}$. So $\det A, \det C \neq 0$ and so the determinant of ϕ is $\det A \det C \neq 0$.

13. Suppose $g = \sum_i a_i f_i$. Then for any $x \in \cap_{i=1}^n \ker f_i$, we have $f_i(x) = 0$ for all i and so $g(x) = 0$. So $x \in \ker g$ and so $\cap_{i=1}^n \ker f_i \subset \ker g$.

For the converse, assume $\cap_{i=1}^n \ker f_i \subset \ker g$. Let $\mathbb{F}$ be the underlying field. Define the function

$$F : V \to \mathbb{F}^n,\ F(v) = \begin{pmatrix} f_1(v) \\ f_2(v) \\ \vdots \\ f_n(v) \end{pmatrix}$$

and extend g to g' where

$$g' : V \to \mathbb{F}^n,\ g'(v) = \begin{pmatrix} g(v) \\ g(v) \\ \vdots \\ g(v) \end{pmatrix}.$$

Then $\ker F = \cap_{i=1}^n \ker f_i \subset \ker g = \ker g'$. By Question 5(b), we have $(\ker g')^\circ \subset (\ker F)^\circ$ and so $Im(g'^*) \subset Im(F^*)$. This shows that there exists θ such that

$$g'^*(proj_1) = F^*(\theta)$$

where $proj_1 : \mathbb{F}^n \to \mathbb{F}$ is the projection onto the first component (and so it is linear). So $g = \theta \circ F$. Since θ is a linear function $\mathbb{F}^n \to \mathbb{F}$, so it can be viewed as a $1 \times n$ matrix. So there exist $a_1, \ldots, a_n$ such that $g = \sum_i a_i f_i$ as a function.

14. (i) If α is singular, then take v which is not in the image of α. So v and $\alpha(v)$ must be linearly independent. Assume α is not singular, then $\mathrm{Tr}(\alpha) = 0$ implies α has at least two distinct eigenvalues say λ and μ, because the sum of eigenvalues is zero and none of the eigenvalues can be zero.

Let $\alpha u = \lambda u$ and $\alpha w = \mu w$. Take $v = u + w$ then $\alpha v = \lambda u + \mu w$. We check $u + w$ and $\lambda u + \mu v$ are linearly independent. Indeed, if $a(u + w) + b(\lambda u + \mu v) = 0$ then $(a + b\lambda)u + (a + b\mu)v = 0$. But u, v are independent because they come from different eigenspace. So $a + b\lambda = a + b\mu = 0$. Since $\lambda \neq \mu$, we have $a = b = 0$. To make sure v is real, either both λ and μ are real in which case u and w are real; or λ is not real in which case we take μ to be the complex conjugate and so we can take $w = \bar{u}$ so that $v = u + w$ is real.

We prove the statement by induction on n, the dimension of V. If $\dim V = 2$ then take v with v and αv linearly independent so they form a basis. Then with respect to this basis the first column of α is $(0, 1)^T$. Then the bottom right entry must be zero because $\mathrm{Tr}(\alpha)$ is independent of the choice of basis. Suppose the statement is true for $n (n \geq 2)$, then for $n + 1$, take v such that v and αv are linearly independent. Extend $\{v, \alpha v\}$ to a basis and so with respect to this basis the matrix of α is

$$\begin{pmatrix} 0 & u^T \\ c & B \end{pmatrix}$$

where $c = (1, 0, \ldots, 0)^T$ and B is some $n - 1 \times n - 1$ matrix.

Consider the linear map on the $n - 1$ dimensional matrix represented by the matrix B. We have $\mathrm{Tr}\, B = 0$ and so by inductive hypothesis we can take a basis so that B is similar to a matrix whose diagonal is zero. Intuitively, the matrix obtained by deleting the first row is the linear map $\alpha : V \to V/v$, and the matrix B can be viewed as a linear map from V/v to itself represented by the entries of B.

Then together with v, we obtain a basis such that α has diagonal entry 0.

(ii) We prove the following statement: Suppose $C = XY - YX$ for some X, Y, then there exist matrices X' and Y' such that

$$\begin{pmatrix} 0 & u^T \\ v & C \end{pmatrix} = X'Y' - Y'X'.$$

If we replace X by $X + \lambda I$ then $(X + \lambda I)Y - Y(X + \lambda I) = XY - YX = C$ so we may assume that X is invertible. Let

$$X' = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix},\ Y' = \begin{pmatrix} 0 & -u^T X^{-1} \\ X^{-1}v & Y \end{pmatrix}$$

and so we have

$$X'Y' - Y'X' = \begin{pmatrix} 0 & 0 \\ v & XY \end{pmatrix} - \begin{pmatrix} 0 & -u^T \\ 0 & YX \end{pmatrix} = \begin{pmatrix} 0 & u^T \\ v & XY - YX \end{pmatrix} = \begin{pmatrix} 0 & u^T \\ v & C \end{pmatrix}.$$

So we prove this by induction. Suppose $\dim V = 2$, then by (i) we can pick a basis so that $\alpha = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ for some a, b. Take

$$X = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ Y = \begin{pmatrix} 0 & -a \\ b & 0 \end{pmatrix}$$

so that $XY - YX = \alpha$. Then the result follows from induction and the above argument.

15. The map $\alpha|_Y$ is just the restriction and since $\alpha(Y) \subset Z$ so it has the correct image. The second map is the quotient. It is well-defined because if $v_1 + Y = v_2 + Y$, i.e. $v_1 - v_2 \in Y$, then $\alpha(v_1) - \alpha(v_2) = \alpha(v_1 - v_2) \in Z$. So $\alpha(v_1) + Z = \alpha(v_2) + Z$. The maps are clearly linear.

    If $i \leq k$ then $\alpha(v_i) \subset Z$ and so $\alpha(v_i) = \sum_{k=1}^{l} a_{ij} w_j$. Therefore the bottom left is a block of zeroes. The matrix for $\alpha|_Y$ is just A and The matrix for $\bar{\alpha}$ is B.