Linear Algebra 3

zc231

1. The first matrix has characteristic polynomial $(x-2)(x-1)^2$. The eigenspace for 2 is generated by $(2,2,1)^T$. The eigenspace for 1 is generated by $(1,0,0)^T$.

   For the second matrix, we again have characteristic polynomial $(x-2)(x-1)^2$. The eigenspace for 2 is generated by $(1,2,1)^T$. The eigenspace for 1 is generated by $(1,1,1)^T$ and $(0,1,1)^T$.

   For the third matrix, the characteristic polynomial is $(x-2)^2(x-1)$. The eigenspace for 2 is generated by $(0,1,1)^T, (1,2,1)^T$. The eigenspace for 1 is generated by $(1,1,1)^T$.

   So we can take $(1,1,1)^T, (0,1,1)^T, (1,2,1)^T$ for the common basis so that the second and the the third matrices are both diagonal.

2. Clearly $(1,1,\ldots,1)^T$ is an eigenvector with eigenvalue $\lambda+n-1$. If $\lambda=1$ then the matrix is singular, and so the determinant, viewed as a polynomial in λ, must contain a factor $\lambda-1$. This means $\lambda-1$ is an eigenvalue. Indeed, the matrix $A-(\lambda-1)I$ is singular and the dimension of the kernel of this is $n-1$. So the dimension of the eigenspace with eigenvalue $\lambda-1$ is $n-1$. Therefore the determinant must be $c(\lambda-1)^{n-1}(\lambda+n-1)$. By considering the coefficient of λ^n, we conclude that $c=1$ and so the determinant is $(\lambda-1)^{n-1}(\lambda+n-1)$.

3. We have $\chi_\alpha(t)=\det(tI-\alpha)$ and so $\chi_\alpha(0)=\det(-\alpha)=(-1)^n\det\alpha$. So $\det\alpha=(-1)^n c_0$. The trace is independent of the choice of the basis. So put α in JNF and so the trace is the sum of eigenvalues, which is $-c_{n-1}$.

4. Since

   $$v = id_V(v) = \pi_1(v) + \cdots + \pi_k(v)$$

   so $V = U_1 + \cdots + U_k$. We check $U_i \cap U_j = \{0\}$ for all $i \neq j$. For each j we have

   $$\pi_j = id_V \pi_j = \pi_1\pi_j + \cdots + \pi_k\pi_j = \pi_j^2.$$

   Suppose $z \in U_i \cap U_j$ and so we have

   $$z = \pi_j(x) = \pi_i(y).$$

   Apply π_j on both sides we have

   $$\pi_j^2(x) = \pi_j\pi_i(y) = 0.$$

   But $\pi_j^2 = \pi_j$ so $z = \pi_j(x) = 0$. This shows that the sum is a direct sum.

   For the second part, define

   $$\pi_1 = \frac{1}{2}(\alpha^2+\alpha),\ \pi_2 = 1-\alpha^2,\ \pi_3 = \frac{1}{2}(\alpha^2-\alpha).$$

   Then $id_V = \pi_1+\pi_2+\pi_3$ and $\pi_i\pi_j = 0$. So $V = U_1 \oplus U_2 \oplus U_3$. But U_1 is the image of π_1. If $x \in Im(\pi_1)$ then $(\alpha-1)x = 0$ and so $x \in V_1$. Conversely, if $x \in V_1$ then $\alpha x = x$. So $\alpha^2 x = \alpha x = x$ and so

   $$x = \frac{\alpha^2+\alpha}{2}x.$$

So $V_1 = U_1$. Similarly, $V_0 = U_2, V_{-1} = U_3$ and so the result follows. Alternatively, it is clear that $U_1 \subset V_1, U_2 \subset V_0$ and $U_3 \subset V_{-1}$ so we have

$$V = U_0 \oplus U_1 \oplus U_2 \subset V_1 \oplus V_0 \oplus V_{-1} \subset V$$

and so equality must hold for each of them.

5. If $\alpha v = \lambda v$ then
$$\alpha^2 v = \alpha(\lambda v) = \lambda^2 v.$$
Let μ be an eigenvector of $\beta = \alpha^2$ and λ a square root of μ. Then
$$0 = \det(\alpha^2 - \mu I) = \det(\alpha - \lambda I)\det(\alpha + \lambda I).$$
So either $\det(\alpha - \lambda I) = 0$ or $\det(\alpha + \lambda I) = 0$.

Not necessarily the same. For example, let
$$\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
and so $\alpha^2 = 0$. The only eigenvalue of α, α^2 is 0. But $\ker(\alpha^2) = \mathbb{R}^2$ whereas $\ker(\alpha) = \langle (1,0)^T \rangle$.

6. Apply question 7 from the first example sheet with $U = V$, $W = Im(\alpha_2) \subset V$ and $\alpha = \alpha_1$. Then $r(\alpha|_W) = r(\alpha_1\alpha_2)$ and so
$$r(\alpha_1\alpha_2) \geq r(\alpha_1) - \dim V + r(\alpha_2).$$
Apply Rank-Nullity so we have
$$n(\alpha_1\alpha_2) \leq n(\alpha_1) + n(\alpha_2).$$
Let $p(x) = (x - \lambda_1)\cdots(x - \lambda_k)$. Since $p(\alpha) = 0$ so λ_i are eigenvalues of α. Apply the first part repeatedly so we have
$$\dim V = n(p(\alpha)) \leq n(\alpha - \lambda_1 I) + \cdots + n(\alpha - \lambda_k I) \leq \dim V.$$
because eigenspaces are disjoint. So the equality must hold and by comparing the dimensions we conclude that
$$V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_k}.$$

7. The minimal polynomial of A divides $x^m - 1$ which has distinct linear factors, and so A can be diagonalised.

8. Clearly the first two are not similar to the third one because if A is similar to the identify matrix then $A = P^{-1}IP = I$. For the first and the second one, both of them have characteristic polynomial $(x-1)^3$. One way to show that they are not similar is that to check they have different minimal polynomial. Indeed the first matrix has minimal polynomial $(x-1)^3$ and the second matrix has minimal polynomial $(x-1)^2$. The fourth matrix is similar to the first one. Let v_1, v_2, v_3 be the basis with respect to which the linear map α represented by the second matrix has
$$\alpha v_1 = v_1,\ \alpha v_2 = v_1 + v_2,\ \alpha v_3 = v_2 + v_3.$$
Now take the basis $u_1 = v_1, u_2 = v_2$ and $u_3 = v_2 + v_3$, and so
$$\alpha u_1 = u_1,\ \alpha u_2 = u_1 + u_2,\ \alpha u_3 = v_1 + 2v_2 + v_3 = u_1 + u_2 + u_3.$$

9. The characteristic polynomial is $(x-1)^2$ and the eigenspace of 1 is generated by $(1,-1)^T$. We want to find v such that $Av = (1,-1)^T + v$. So we can just take v to be $(0,-1)^T$. So we take the basis $\{(1,-1)^T, (0,-1)^T\}$. We have $PAP^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ where $P = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$. So

$$A^n = P^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n P = P^{-1} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} P = \begin{pmatrix} 1-n & -n \\ n & n+1 \end{pmatrix}.$$

10. (a) Consider

$$M_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ M_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then both of them have characteristic polynomial $(x-1)^4$ and minimal polynomial $(x-1)^2$.

(b) Since $A^2 \neq A$ so the minimal polynomial is not $x^2 - x$. But $A^4 - A^2 = 0$ and so the minimal polynomial is a factor of $x^4 - x^2 = x^2(x^2-1)$. So the minimal polynomial can be $x^4 - x^2, x^3 - x, x^2 - 1, x^3 + x^2, x^2 + x, x + 1$ or $x^3 - x^2$. For the characteristic polynomials, we basically take the same linear factor of the minimal polynomial and raise those into some power so that the sum of the indices is equal to 5. For example, if the minimal polynomial is $x^3 - x = x(x-1)(x+1)$ then the possible characteristic polynomial is $x^m(x-1)^n(n+1)^r$ where $m+n+r=5$.

If A is not diagonalisable then the JNF J_A must contain a block of size at least 2. Take P such that $P^{-1}J_AP = A$. Then $A^4 = A^2 \neq A$ if and only if $J_A^4 = J_A^2 \neq J_A$. So this means that if B is a block in J_A, we must have $B^4 = B^2$. We consider the possible block.

If $B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ then $B^2 = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}$ and $B^4 = \begin{pmatrix} \lambda^4 & 4\lambda^3 \\ 0 & \lambda^4 \end{pmatrix}$. So we must have $\lambda = 0$. If B has size bigger than 2, then we can check that it is impossible to have $B^4 = B^2$. Therefore, we conclude that any Jordan block in J_A must have size 1 or 2. In particular, if B is the above block, we have $B^4 = B^2$ and $0 = B^2 \neq B$. We can either have two blocks of size 2 plus one block of size 1; or one block of size 2 and three blocks of size 1. In the former case, we have 3 cases (the remaining entry can be 0 or ± 1) and in the later case, we have 10 cases (up to similarity) so in total we have 13 cases.

11. Take y such that $\alpha^{n-1}y \neq 0$. Suppose there exist $a_1, \dots, a_n$, not all zero, such that $\sum_i a_i \alpha^{i-1} y = 0$. Let k be the smallest integer such that $a_k \neq 0$ and so we have $\sum_{i=k}^n a_i \alpha^{i-1} y = 0$. Apply α^{n-k} to the equation, and since $\alpha^n = 0$, we get $a_k \alpha^{n-1} y = 0$. But $\alpha^{n-1} y \neq 0$ so $a_k = 0$ which is a contradiction. So $a_i = 0$ for each i and so $\{y, \alpha y, \dots, \alpha^{n-1} y\}$ is a basis.

Let $\beta y = \sum_i a_i \alpha^{i-1} y = p(\alpha) y$ for some polynomial α. Then for each j,

$$(\beta - p(\alpha))\alpha^j y = (\beta\alpha^j)(y) - p(\alpha)\alpha^j(y) = \alpha(\beta - p(\alpha))(y) = 0.$$

But $\{\alpha^j y : j = 0, \dots, n-1\}$ is a basis, so $\beta = p(\alpha)$. The first column of β is $(a_0, a_1, \dots, a_n)^T$. Since

$$\beta\alpha y = \alpha\beta y = \alpha \sum_i a_i \alpha^{i-1} y = \sum_i a_i \alpha^i y$$

so the second column is $(0, a_1, a_2, \dots, a_{n-1})^T$. So similarly, the ith column would just be $(0, 0, \dots, a_1, \dots, a_{n-i+1})^T$.

12. λ is an eigenvalue of α^{-1} if and only if there exists v such that $\alpha^{-1} v = \lambda v$, if and only if $\alpha v = \lambda^{-1} v$. So the eigenvalues of α^{-1} are of the form λ where λ is an eigenvalue of α.

Let $f(t)$ be the characteristic polynomial of α and $g(t)$ be the characteristic polynomial of α^{-1}. Then

$$\det(tI - \alpha^{-1}) = \det(-\alpha^{-1})\det(-\alpha tI + I) = \det(-\alpha^{-1})t^n \det(t^{-1}I - \alpha) = \det(-\alpha^{-1})t^n f(1/t).$$

Similarly, α is a root of $t^n + a_{n-1}t^{n-1} + \cdots + a_0 = 0$, if and only if α^{-1} is a root of $1 + a_{n-1}t + \cdots + a_0 t^n = 0$. Therefore, if $f(t)$ is the minimal polynomial of α, then the minimal polynomial of α^{-1} is $t^n f(1/t)/a0$ (because minimal polynomial is monic).

13. α and α^{-1} share the same eigenvectors and so the dimensions of the eigenspaces of α are equal to those of α^{-1}. If α is represented by the Jordan block $J_m(\lambda)$, then α^{-1} has only one eigenvalue λ with geometric multiplicity 1. Therefore, the Jordan Normal Form of α^{-1} is $J_m(\lambda^{-1})$. Therefore if A has Jordan blocks $J_{n_i}(\lambda_i), i = 1, \ldots, k$ then A^{-1} has JNF with Jordan blocks $J_{n_i}(\lambda^{-1})$.

    For the second part, it suffices to prove the statement for a complex matrix in JNF. So it suffices to prove that a complex Jordan block is similar to its transpose. Let α be a Jordan block with eigenvalue λ. Then we have a basis $v_1, \ldots, v_k$ such that

$$\alpha v_1 = \lambda v_1,\ \alpha v_2 = v_1 + \lambda v_2, \ldots, \alpha v_k = v_{k-1} + \lambda v_k.$$

    Then let $u_1 = v_k, u_2 = v_{k-1}, \ldots, u_k = v_1$. Then with respect to this basis, the matrix of α is the transpose of $J_k(\lambda)$.

14. $f(\lambda) = \det(A + \lambda B)$ is a polynomial in λ (of finite degree) and $f(\lambda) \not\equiv 0$ because $f(i) \neq 0$. So there are only finitely many values of λ such that $A + \lambda B$ is NOT invertible. So we can pick a real number λ such that $A + \lambda B$ is invertible.

    If P and Q are similar over $\mathbb{C}$, then there exists $C = A + iB$ such that $CPC^{-1} = Q$ and so $CP = QC$. This gives

$$(A + iB)P = Q(A + iB).$$

    Since P, Q are both real matrices, by comparing real and imaginary parts on both sides we have $AP = QA$ and $BP = QB$. Take λ such that $A + \lambda B$ is invertible and then we have

$$(A + \lambda B)P = Q(A + \lambda B)$$

    which is equivalent to $(A + \lambda B)P(A + \lambda B)^{-1} = Q$.

15. Let $v_0, v_1, \ldots, v_n$ be the columns of the matrix. Replace v_0 by $v_0 + v_1 x + v_2 x^2 + \cdots + v_n x^n$ where $x = \zeta^j$ for some j. Then the kth entry of the first column becomes $x^{k-1}f(x)$, using $x^{n+1} = 1$. This is true for all j and so $f(\zeta^j)$ is a factor of the discriminant. Since $f(\zeta^j), j = 0, 1, \ldots, n$ are all distinct, so $\prod_{j=0}^{n} f(\zeta^j)$ is a factor of the discriminant. Now compare the degree of a_0 in the discriminant, we conclude that $\prod_{j=0}^{n} f(\zeta^j)$ is the discriminant.

16. (i) Let $f_\lambda(x) = e^{\lambda x}$ then $\alpha f_\lambda = \lambda f_\lambda$ and so λ is an eigenvalue of α with eigenvector f_λ. Suppose now $g(x)$ is a function in $\ker(\alpha - \lambda I)$, then $g'(x) = \lambda g(x)$, which gives $g(x) = ce^{\lambda x}$ for some constant c and so $g(x) \in \langle f_\lambda \rangle$.

    (ii) We need to show that for all $g(x)$ there exists $f(x)$ such that $f'(x) - \lambda f(x) = g(x)$. But this is just first order ODE and it has a general solution so such $f(x)$ must exist.