Linear Algebra 4

zc231

1. Let A_1, A_2, A_3, A_4 be the given matrices which all have rank 2 and so they are congruent to I over $\mathbb{C}$. A_1, A_4 have two positive eigenvalues and so they are congruent to I over $\mathbb{R}$.

   A_3 has two negative eigenvalues so it is not congruent to I over $\mathbb{R}$. Similarly, A_2 is congruent to I over $\mathbb{R}$ because it has one negative eigenvalue.

   If $P^T AP = I$ then $\det A = 1/(\det P)^2$ is a square. Since A_1 has non-square determinant (over $\mathbb{Q}$) so it is not congruent to I over $\mathbb{Q}$. For A_4, we observe that the following row and column operations transforms A_4 into I

$$A_4 \mapsto \begin{pmatrix} 4 & 4 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mapsto I.$$

   So let $P = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}$ gives $P^T A_4 P = I$.

2. For the first one,

$$x^2 + y^2 + z^2 - 2xz - 2yz = (x-z)^2 + (y-z)^2 - z^2$$

   and so the rank is 3 and the signature is 1. On the other hand, we have

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

   and so by row operations and column operations we have $P^T AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ where $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

   For the second one, we have

$$x^2 + 2y^2 - 2z^2 - 4xy - 4yz = (x-2y)^2 - 2(y+z)^2$$

   and so the rank is 2 and the signature is 0.

   For the third one, we have

$$16xy - z^2 = (2x+2y)^2 - (2x-2y)^2 - z^2$$

   and so the rank is 3 and the signature is -1.

   Finally for the last one, by considering the associated matrix, we conclude that the rank is 3. since it has two negative eigenvalues and one positive eigenvalue so the signature is -1. Alternative, you can try to complete the squares:

$$2xy + 2yz + 2xz = \frac{1}{2}(x+y+2z)^2 - \frac{1}{2}(x-y)^2 - 2z^2.$$

3. (i) $\psi(A,B) = \psi(B,A)$ because

$$\mathrm{tr}(AB^T) = tr((AB)^T) = tr(B^TA) = tr(AB^T).$$

It is bilinear because trace is linear. It is positive definite because

$$\mathrm{tr}(AA^T) = \sum_i \sum_j A_{ij}A_{ij} = \sum_{i,j} A_{ij}^2 \geq 0$$

and is 0 if and only if $A_{ij} = 0$ for all i,j, in which case $A = 0$. The inequality follows from Cauchy-Schwarz.

(ii) Let $Q(A) = \mathrm{tr}(A^2)$. Then $Q(\lambda A) = \mathrm{tr}(\lambda^2 A^2) = \lambda^2 Q(A)$. For any A, B,

$$(A,B) \mapsto Q(A+B) - Q(A) - Q(B) = \mathrm{tr}(AB) + \mathrm{tr}(BA) = 2\,\mathrm{tr}(AB)$$

is bilinear because the trace is linear. The associated bilinear map ϕ has entries

$$\phi(A,B) = \frac{1}{2}(Q(A+B) - Q(A) - Q(B)) = \mathrm{tr}(AB) = \psi(A, B^T).$$

Now if $A^T = A$ then $\phi(A,A) = \psi(A,A) > 0$ and if $A^T = -A$ then $\phi(A,A) = \psi(A,-A) < 0$ where ψ is the map in (i).

Every matrix can be written as a sum of symmetric and antisymmetric matrix

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}.$$

Therefore, if X is the subspace of symmetric matrices and Y is the subspace of anti-symmetric matrices then $X \oplus Y = Mat_n(\mathbb{R})$. Also the previous paragraph shows that ϕ is positive definite on X and is negative definite on Y. Therefore, if $p+q$ and $p-q$ represent the rank and signature of ϕ respectively, we must have $\dim X \leq p$ and $\dim Y \leq q$. But $p + q \leq n^2$ and $\dim X + \dim Y \leq n^2$, so we must have $p = \dim X$ and $q = \dim Y$.

Finally, $\dim X = \binom{n}{2} + n = \frac{n(n+1)}{2}$ and $\dim Y = \binom{n}{2} = \frac{n(n-1)}{2}$. So the rank of Q is n^2 and the signature is n.

4. The form corresponds to the matrix

$$A = \begin{pmatrix} 1 & 1-i & 1+i \\ 1+i & 1 & 1-i \\ 1-i & 1+i & 1 \end{pmatrix}$$

which is congruent to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and so the rank is 3 and the signature is 1.

For the second part, we have

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} \zeta^k \psi(u + \zeta^k v, u + \zeta^k v) &= \frac{1}{n}\sum_{k=1}^{n} \zeta^k(\psi(u,u) + \psi(v,v)) + \frac{1}{n}\sum_{k=1}^{n} \zeta^{2k}\psi(v,u) + \frac{1}{n}\sum_{k=1}^{n} \psi(u,v) \\ &= \frac{\zeta^2(1-\zeta^{2n})}{1-\zeta^2}\psi(v,u) + \psi(u,v) = \psi(u,v) \end{aligned}$$

because $n > 2$ and so $\zeta^2 \neq 1$.

5. Completing squares we have

$$2(x^2+y^2+z^2+xy+yz+xz) = (x+y)^2+(y+z)^2+(x+z)^2$$

and so it is positive definite. Let

$$X = x+y,\ Y = y+z,\ Z = x+z$$

so that

$$x = \frac{X+Z-Y}{2},\ y = \frac{X+Y-Z}{2},\ z = \frac{-X+Y+Z}{2}.$$

This gives an Orthonormal basis

$$e_1 = \frac{1}{2}(1,-1,1)^T,\ e_2 = \frac{1}{2}(1,1,-1)^T,\ e_3 = \frac{1}{2}(-1,1,1)^T.$$

Let $\{v_1, v_2, v_3\}$ be the standard basis. So $e_1 = v_1/\sqrt{2} = (1/\sqrt{2}, 0, 0)^T$. Let

$$f_2 = v_2 - \langle e_1, v_2\rangle e_1 = v_2 - \frac{1}{2}v_1 = (-\frac{1}{2}, 1, 0)^T$$

and so $e_2 = \sqrt{\frac{2}{3}}(-1/2, 1, 0)^T$. Let

$$f_3 = v_3 - \langle e_1, v_3\rangle e_1 - \langle e_2, v_3\rangle e_2 = v_3 - \frac{1}{2}v_1 - (-1/6, 1/3, 0)^T = 1/3(-1,-1,3)^T$$

and so $e_3 = \frac{1}{2\sqrt{3}}(-1,-1,3)^T$.

6. Let $V = W \oplus W^\perp$. For each $v \in V$, write $v = u + w$ where $u \in W^\perp$ and $w \in W$. The orthogonal projection is $\pi : V \to W$ sending v to w. So $\pi^2(u+w) = \pi(w) = w$ and π is an idempotent. For each $v_1, v_2 \in V$, let $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$. So we have

$$\langle \pi(v_1), v_2\rangle = \langle w_1, u_1 + w_2\rangle = \langle w_1, w_2\rangle = \langle u_1 + w_1, w_2\rangle = \langle v_1, \pi(v_2)\rangle$$

and so π is self-adjoint.

Conversely, if π is self-adjoin and idempotent, then $\ker\pi \oplus Im(\pi) = V$ (see sheet 1 question 5). We will show that $\ker\pi = Im(\pi)^\perp$. We have $u \in \ker\pi$ if and only if $\pi u = 0$, if and only if $\langle v, \pi u\rangle = 0$ for all v (using that the inner product is positive definite so we can pick $v = \pi u$), if and only if $\langle \alpha v, u\rangle = 0$ for all v, if and only if $u \in Im(\alpha)^\perp$.

7. Since S is real symmetric, every eigenvalue of S is real and there exists orthogonal matrix P such that $P^TSP = D$ where D is a diagonal matrix whose entries are eigenvalues of S. So $S^k = I$ implies $D^k = I$ and so the eigenvalues of S are kth roots of unity. But the only real roots of unity are ± 1 and so $D^2 = I$. So $S^2 = I$.

8. (i) α is self-adjoint, and so we can take an orthonormal basis such that the matrix A of α with respect to this basis is Hermitian. Let λ be an eigenvalue such that $\alpha v = \lambda v$. Then $\lambda\langle v, v\rangle = \langle \alpha v, v\rangle > 0$ and so $\lambda > 0$. This shows that every eigenvalue is positive.

   Take an orthonormal basis such that the matrix A is diagonal whose entries are eigenvalues of A. Let B be the matrix whose entries are positive square roots of the eigenvalues of A, with respect to this basis. Then B is positive definite and $B^2 = A$.

   On the other hand, if μ is an eigenvalue of B with eigenvector v where $B^2 = A$, then $\mu^2 = \lambda$ is an eigenvalue of A with eigenvector A. So by picking the same orthonormal basis, the the entries of B are determined by the eigenvalues of A.

(ii) We have $\langle \alpha^*\alpha v, v\rangle = \langle \alpha v, \alpha v\rangle > 0$ for all non-zero v (because α has trivial kernel). Also,

$$\langle \alpha^*\alpha v, v\rangle = \langle v, \alpha^*\alpha v\rangle$$

and so $\alpha^*\alpha$ is positive definite. By (i) let $B^2 = \alpha^*\alpha$ and so

$$\alpha = (\alpha^*)^{-1}B^2.$$

Let $\beta = (\alpha^*)^{-1}B$ and $\gamma = B$ so that γ is positive definite by (i). We have

$$\beta^{-1} = B^{-1}\alpha^* = B(B^2)^{-1}\alpha^* = B(\alpha^*\alpha)^{-1}\alpha^* = B\alpha^{-1} = \beta^*$$

(because $B = B^*$). Therefore, β is unitary.

9. Let λ be an eigenvalue of α and let U be the eigenspace of λ. For each $u \in U$, we have

$$\alpha(\alpha^*u) = \alpha^*(\alpha u) = \alpha^*(\lambda u) = \lambda(\alpha^*u)$$

and so $\alpha^*u \in U$. So $\alpha^*(U) \subset U$. So we can consider the restriction $\alpha^*|_U$, which is an endomorphism of U. Let v be an eigenvalue of this restriction, and so $\alpha^*v = \mu v$ for some μ. But $v \in U$ and so $\alpha v = \lambda v$. Since

$$\mu\langle v, v\rangle = \langle \alpha^*v, v\rangle = \langle v, \alpha v\rangle = \bar{\lambda}\langle v, v\rangle$$

we conclude that $\mu = \bar{\lambda}$.

Let $u \in \langle v\rangle^{\perp}$. Then

$$\langle v, \alpha u\rangle = \langle \alpha^*v, u\rangle = \mu\langle v, u\rangle = 0$$

and

$$\langle v, \alpha^*u\rangle = \langle \alpha v, u\rangle = \lambda\langle v, u\rangle = 0$$

and so $\alpha(u)$ and $\alpha^*(u)$ are both contained in $\langle v\rangle^{\perp}$.

Finally, we start with $v_1 = v$ and let $V_1 = \langle v_1\rangle$. Consider the restriction of α and α^* on $V_1^{\perp}$. Then there is an eigenvector $v_2 \in V_1$ of α and α^*. Let $V_2 = \langle v_1 \oplus v_2\rangle^{\perp}$. Then $V = V_2 \oplus V_2^{\perp}$ and we consider the restriction of α and α^* on $V_2^{\perp}$. We can repeat this to get a set of orthogonal basis (and then normalise the vectors).

10. Let

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 3 & -3 \\ 3 & 3 & 1 \\ -3 & 1 & 3 \end{pmatrix}.$$

We first put A into diagonal form. Clearly, using row and column operations, we find that if

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & \sqrt{3}/2\sqrt{2} \end{pmatrix}$$

then $P^TAP = I$ and

$$C = P^TBP = \begin{pmatrix} 1/2 & \sqrt{3}/\sqrt{2} & -\sqrt{3}/2 \\ \sqrt{3}/\sqrt{2} & 1 & 1/\sqrt{2} \\ -\sqrt{3}/2 & 1/\sqrt{2} & 3/2 \end{pmatrix}.$$

Since $C = P^TBP$ is also symmetric (this is why the method works), there exists an orthogonal matrix Q such that Q^TCQ is diagonal. Also $Q^TIQ = I$. So if we find such Q, we must have $(PQ)^T A(PQ) = I$ and $(PQ)^T B(PQ)$ is diagonal. Indeed, in this case the characteristic polynomial of C is $(t-2)^2(t+1)$ and we can take

$$Q = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & -1/\sqrt{6} & \sqrt{2}/\sqrt{3} \end{pmatrix}$$

so that $Q^T CQ = diag(-1, 2, 2)$.

Alternatively, consider $C = A^{-1}B$. We have

$$C = \begin{pmatrix} 1/2 & 3/2 & -3/2 \\ 3/4 & 5/4 & 3/4 \\ -3/4 & 3/4 & 5/4 \end{pmatrix}.$$

We can diagonalise C and we find that $P^{-1}CP$ is diagonal where

$$P = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \text{ and } P^{-1}CP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then we have

$$P^T AP = \begin{pmatrix} 5 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 16 \end{pmatrix} \text{ and } P^T BP = \begin{pmatrix} 10 & 6 & 0 \\ 6 & 10 & 0 \\ 0 & 0 & -16 \end{pmatrix}.$$

So we now need to make the top left 2×2 submatrices simultaneously diagonal. But this is clear because they are scaler multiple to each other. Let

$$Q = \begin{pmatrix} 1 & -3/5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we have $(PQ)^T APQ = diag(5, 16/5, 16)$ and $(PQ)^T P^T APQ = diag(10, 32/5, -16)$. Finally, rescale the matrix PQ by $diag(1/\sqrt{5}, \sqrt{5}4, 1/4)$ so we obtain $X^2+Y^2+Z^2$ and $2X^2+2Y^2-Z^2$.

Following a similar strategy, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We compute

$$C = A^{-1}B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $P^{-1}CP$ is diagonal where

$$P = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}.$$

Then we have

$$P^T AP = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } P^T BP = \begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix}.$$

11. A represents a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Since the rank is n, so the kernel of A is trivial by Rank-Nullity. Suppose $A^T Av = 0$, then

$$(Av)^T Av = v^T A^T Av = 0.$$

Since Av is a real vector, we must have $Av = 0$ and hence $v = 0$. So the kernel of $A^T A$ is trivial and so it is invertible. For complex matrix, replace A^T by $A^\dagger$.

12. Clearly $(,)$ is symmetric. Integration is a linear operator, so $(,)$ is bilinear. Finally,

$$(f, f) = \int_{-1}^{1} f^2(t)dt \geq 0$$

because $f^2(t)$ is non-negative and continuous and equality holds if and only if $f(t) = 0$ inside $[-1, 1]$, if and only if $f(t) \equiv 0$. So it is positive definite and hence an inner product.

Using integration by parts, we have

$$\begin{aligned}
\langle \alpha f, g \rangle &= \int_{-1}^{1} (1-t^2)f''(t)g(t)dt - \int_{-1}^{1} 2tf'(t)g(t)dt \\
&= \int_{-1}^{1} 2tg(t)f'(t)dt - \int_{-1}^{1} (1-t^2)g'(t)f'(t)dt - \int_{-1}^{1} 2tg(t)f'(t)dt \\
&= \int_{-1}^{1} -2tg'(t)f(t)dt + \int_{-1}^{1} (1-t^2)g''(t)f(t)dt = (f, \alpha g).
\end{aligned}$$

So α is self-adjoint.

Let λ be an eigenvalue of α with eigenvector f. Then $\alpha f = \lambda f$ and so we have

$$(1-t^2)f''(t) - 2tf'(t) - \lambda f(t) = 0.$$

Consider the leading term of f, say t^m (the scaler does not matter because the equation is linear in f). We must have

$$-m(m-1) - 2m - \lambda = 0$$

and so the eigenvalues of $-m^2 - m$ where $m = 0, 1, \ldots, n$.

(i) For each $n < k$ we have $\frac{d^m}{dt^m}(1-t^2)^k = 0$ at $t = \pm 1$. Let $i < j$. We have

$$(s_i, s_j) = \int_{-1}^{1} \frac{d^i}{dt^i}(1-t^2)^i \frac{d^j}{dt^j}(1-t^2)^j dt = (-1)^j \int_{-1}^{1} \frac{d^{i+j}}{dt^{i+j}}(1-t^2)^i (1-t^2)^j dt = 0$$

by using integration by parts j times.

(ii) By (i) we know these elements are linearly independent so it forms a basis.

(iii) $s_k \in P_k$ and by (i) it spans the orthogonal complement of P_{k-1}, because by (ii) $s_0, \ldots, s_{k-1}$ is a basis for P_{k-1}.

(iv) Induction on k. For $k = 0$ we have $s_0 = 1$ and so s_0 is an eigenvector of α with eigenvalue 0. Suppose s_i is an eigenvector of α with eigenvalue λ_i, then by (i), $(s_{i+1}, s_j) = 0$ for all $j \leq i$. Then

$$(\alpha s_{i+1}, s_j) = (s_{i+1}, \alpha s_j) = \lambda(s_{i+1}, s_j) = 0$$

and so αs_{i+1} is in orthogonal complement of P_i. So $\alpha s_{i+1} = \lambda_{i+1} s_{i+1}$ for some λ_{i+1} by (iii). This shows that s_k is an eigenvector for all k. In particular, the leading term of s_k has degree k and so the corresponding eigenvalue is $-k^2 - k$.

Let $\{e_0, e_1, \ldots, e_n\}$ be the basis obtained by Gram-Schmidt. Then s_k is a multiple of e_k for each k. First note that s_k and e_k both have degree k for each k and $\{e_0, \ldots, e_k\}, \{s_0, \ldots, s_k\}$ are bases for P_k. Since s_k spans the orthogonal complement of P_{k-1}, and $(e_k, e_j) = 0$ for all $j < k$ by construction, we conclude that $e_k \in \langle s_k \rangle$ and so e_k is a multiple of s_k.

13. The map $(x, y) \mapsto Q(x+y) - Q(x) - Q(y)$ is bilinear because $f_1, \ldots, f_{t+u}$ are linear and $Q(x+y) - Q(x) - Q(y)$ is

$$\begin{aligned}\sum_{i=1}^{t}(f_i(x+y)^2 - f_i(x)^2 - f_i(y)^2) - \sum_{j=1}^{u}(f_{t+j}(x+y)^2 - f_{t+j}(x)^2 - f_{t+j}(y)^2) \\ = \sum_{i=1}^{t} 2f_i(x)f_i(y) - \sum_{j=1}^{u} 2f_{t+j}(x)f_{t+j}(y)\end{aligned}$$

So Q defines a quadratic form.

Let n be the dimension of the vector space and p and q are integers such that $p+q$ is the rank of Q and $p-q$ is the signature. Let s be the dimension of the subspace

$$K = \{x : f_i(x) = 0 \text{ for all } i\}.$$

Then K is inside the kernel and so $p + q + s \leq n$ by Rank-Nullity.

Let U and W be subspaces of V and $\dim U = n - 1$. Then

$$n = \dim V \geq \dim(U + W) = \dim U + \dim W - \dim U \cap W$$

gives

$$\dim(U \cap W) \geq \dim W - 1.$$

Since $f_1, \ldots, f_{t+u}$ are linear functions, and we can assume they are non-zero (because they do not need to appear in the sum if they are zero). Then, by Rank-Nullity, we conclude that the kernel of f_i has dimension $n-1$, for each i. Apply the inequality in the previous paragraph repeatedly, we have $\dim(\cap_{j=1}^{u} \ker f_{t+j}) \geq n - u$ and $\dim(\cap_{j=1}^{t} \ker f_j) \geq n - t$. Therefore, the dimension of

$$K_1 = \{x : x \in \cap_{j=1}^{u} \ker f_{t+j}, x \notin K\}$$

is at least $n - u - s$ and the dimension of

$$K_2 = \{x : x \in \cap_{j=1}^{t} \ker f_j, x \notin K\}$$

is at least $n - t - s$.

Now if $x \in K_1$, then $Q(x) > 0$. So K_1 is a positive definite subspace of Q and so its dimension is bounded above by p. So we have

$$n - u - s \leq p \leq n - q - s$$

which gives $q \leq u$.

Similarly, K_2 is a negative subspace of Q and so its dimension is bounded above by q. So we have

$$n - t - s \leq q \leq n - p - s$$

which gives $p \leq t$.

14. For each symmetric matrix A, take an orthogonal matrix P such that $PAP^T = D$ is diagonal. For each unit vector v, we have

$$v^T A v = v^T P^T D P v = (PT)^T D(Pv).$$

Since P is orthogonal, so Pv is also a unit vector. Therefore, the maximum of $v^T A v$ is the largest eigenvalue of A and it is achieved if v is the corresponding (unit) eigenvector.

For the quadratic form we have here, the corresponding symmetric matrix A is

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

Clearly, $(1, 1, \ldots, 1)^T$ is an eigenvector of A with eigenvalue 2. But this vector is not contained in the subspace $\sum a_i = 0$.

Let ζ be a primitive nth root of unity. For each $j < n$, the vector $(1, \zeta^j, \zeta^{2j}, \ldots, \zeta^{(n-1)j})^T$ is an eigenvector of eigenvalue $\zeta^j + \zeta^{(n-1)j}$, which is $\cos(2j\pi/n)$. So we have obtained all the eigenvalues and clearly these eigenvectors are contained in the linear subspace $\sum_i a_i = 0$. So the maximum is the second largest eigenvalue, $\cos(2\pi/n)$.

15. Let λ be an eigenvalue of α. If λ is real then α has an invariant subspace of dimension 1. If λ is not real, let $\alpha v = \lambda v$ and so $\alpha \bar{v} = \bar{\lambda}\bar{v}$. Then the subspace generated by $\frac{v+\bar{v}}{2}, \frac{(v-\bar{v})}{2}$ is α invariant. Indeed, we can write

$$\alpha(v + \bar{v}) = \lambda v + \bar{\lambda}\bar{v} = (\lambda + \bar{\lambda})(v + \bar{v})/2 + i(\lambda - \bar{\lambda})\frac{v - \bar{v}}{i}.$$

So now we pick any 1 or 2-dimension invariant subspace U and we have $V = U \oplus U^{\perp}$. Let $w \in U^{\perp}$, and so $\langle u, w \rangle = 0$ for all $u \in U$. We have

$$\langle u, \alpha w \rangle = \langle \alpha^{-1} u, w \rangle = 0$$

because α^{-1} also fixes U. Therefore, $U^{\perp}$ is an invariant subspace of α. Now repeat this (inductive argument) so we can decompose V into a direct sum.

If V has dimension 1, then α is a linear map from $\mathbb{R} \to \mathbb{R}$, and so we have $\alpha = \pm id_V$. If V has dimension 2, then we have $\alpha = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ (complex eigenvalues) or $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\alpha = \pm id_V$ (real eigenvalues).