Number Theory 3

zc231

1. Since $\Re(s) > 1$ so the series converges absolutely and so
$$\zeta^2(s) = \prod_{m=1} \frac{1}{m^s} \prod_{l=1} \frac{1}{l^s} = \prod_{m,l=1} \frac{1}{(ml)^s} = \prod_{n=1} \frac{\sum_{m|n} m}{n^s} = \prod_{n=1} \frac{d(n)}{n^s}.$$
For $\Re(s) > 1$ we have
$$\frac{1}{\zeta(s)} = \prod_p (1 - p^{-s}) = \sum_{n=1} \frac{\mu(n)}{n^s}$$
using Euler's product and so
$$\zeta(s-1)/\zeta(s) = \sum_{m=1} \frac{m}{m^s} \sum_{l=1} \frac{\mu(l)}{l^s} = \sum_{m,l=1} \frac{m\mu(l)}{(ml)^s} = \sum_{n=1} \frac{\sum_{m|n} m\mu\left(\frac{n}{m}\right)}{n^s} = \sum_{n=1} \frac{\phi(n)}{n^s}$$
using Mobius inversion (Question 3(ii)) on $n = \sum_{d|n} \phi(d)$.

2. By definition of $\phi(n)$ and $d(n)$ we have
$$\phi(n) = nd(n) - \sigma(n) = n \sum_{m|n} 1 - \sum_{m|n} m = \sum_{m|n} (n-m) = n - 1 + \sum_{m|n, m>1} (n-m)$$
but $\phi(n) \leq n - 1$ for $n > 1$ so this only holds if $1, n$ are the only factors of n, i.e. n is a prime. We should also check $n = 1$ does not work as by convention $\phi(1) = 1$.

3. (i) Since μ is multiplicative, so is $\sum_{d|n} \mu(d)$. Let $n = p^a$ for some prime p and $a \geq 1$. Then
$$\sum_{d|p^a} \mu(d) = 1 + \mu(p) + \cdots + \mu(p^a) = 1 - 1 + 0 + \cdots = 0.$$
Also when $n = 1$, $\sum_{d|1} \mu(d) = 1$ so we conclude the sum is 1 if $n = 1$ and 0 otherwise.
(ii) $f(n) = \sum_{d|n} g(d)$. To see this
$$\sum_{d|n} g(d) = \sum_{d|n} \sum_{m|d} \mu(m) f\left(\frac{d}{m}\right) = \sum_{d|n} \sum_{m_1 m_2 = d} \mu(m_1) f(m_2) = \sum_{m_2|n} f(m_2) \sum_{m_1|\frac{n}{m_2}} \mu(m_1)$$
and by (i) $\sum_{m_1|\frac{n}{m_2}} \mu(m_1) \neq 0$ if and only if $\frac{n}{m_2} = 1$ in which case $m_2 = n$ and so we have $f(n)$.
(iii) Use Mobius inversion formula on $n = \sum_{d|n} \phi(d)$ so $\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$.

4. Let $n = p_1^{a_1} \cdots p_k^{a_k}$. For each $d|n$, $\Lambda(d) = 0$ if d is not a power of p_i for some i so we have
$$\sum_{d|n} \Lambda(d) = \sum_{i=1}^{k} \sum_{j=1}^{a_i} \Lambda(p^{p_i^j}) = \sum_{i=1}^{k} \sum_{j=1}^{a_i} \log p_i = \sum_{i=1}^{k} \log p_i^{a_i} = \log \prod_{i=1}^{k} p_i^{a_i} = \log n.$$

5. 46.

6. (i) For each k, the number of integers less than or equal to N which are divisible by p^k is $\lfloor \frac{N}{p^k} \rfloor$. Then the number of integers which are divisible by p^k but not divisible by p^{k+1} is $\lfloor \frac{N}{p^k} \rfloor - \lfloor \frac{N}{p^{k+1}} \rfloor$. The exact order of p dividing these integers is k. Therefore, the exact order of p dividing $N!$ is

$$\sum_{k=1}^{\infty} \left(\lfloor \frac{N}{p^k} \rfloor - \lfloor \frac{N}{p^{k+1}} \rfloor \right) k = \sum_{k=1}^{\infty} k \lfloor \frac{N}{p^k} \rfloor - \sum_{k=1}^{\infty} \lfloor \frac{N}{p^{k+1}} \rfloor k = \sum_{k=1}^{\infty} k \lfloor \frac{N}{p^k} \rfloor - \sum_{k=2}^{\infty} \lfloor \frac{N}{p^k} \rfloor (k-1) = \sum_{k=1}^{\infty} \lfloor \frac{N}{p^k} \rfloor.$$

(ii) $e^N = 1 + N + \frac{N^2}{2} + \cdots + \frac{N^N}{N!} + \cdots > \frac{N^N}{N!}$ so $N! > \left(\frac{N}{e}\right)^N$.

(iii) It is the equivalent to prove that

$$\prod_{p \le N} p^{\frac{N}{p-1}} > \left(\frac{N}{e}\right)^N.$$

By (i) the left hand side is bigger than $N!$ as $\frac{N}{p-1} > \sum_{k=1} \left[\frac{N}{p^k}\right]$. Then by (ii) the result follows.

7. Let f be a non-constant polynomial with integer coefficients. Suppose f only takes finitely many composite numbers say $n_1, \ldots, n_k$. Let $f(a) = n_1$. Since $b - a | f(b) - f(a)$ for any b so picking $b = a + cn_1$ we have $n_1 | f(a + cn_1) - f(a)$ and so $n_1 | f(a + cn_1)$. There are only finitely many c such that $f(a + cn_1) = n_j$ for some j so take c with $f(a + cn_1) \neq n_j$ for any j and so $f(a + cn_1)$ is a composite number (divisible by n_1) which is not in the list.

8. We prove the following inductive statement. If every integer less than or equal to $2p_k + 6$ can be written as sum of distinct primes less than or equal to p_k then every integer less than or equal to $2p_{k+1} + 6$ can be written as sum of distinct primes less than or equal to p_{k+1} where p_k is the kth prime bigger than 11 (so $p_1 = 13$). So we consider y with $2p_k + 6 < y \le 2p_{k+1} + 6$, then $y - p_{k+1} \le p_{k+1} + 6 < 2p_k + 6$ by Berstrand's postulate and $y - p_{k+1} \ge 2p_k + 6 - p_{k+1} > 6$, again by Berstrand's postulate. So by assumption $y - p_{k+1}$ can be as sum of distinct primes less than or equal to p_k and so we have proved our claim. So to finish the proof we need to check the case for $k = 1$, i.e. writing every integer less than or equal to 32 as sum of distinct primes less than or equal to 13.

$$\begin{aligned}
&7{,}8 = 3 + 5, 9 = 2 + 7, 10 = 3 + 7, 11, 12 = 5 + 7, 13, 14 = 2 + 5 + 7, 15 = 3 + 5 + 7,\\
&16 = 3 + 13, 17 = 2 + 3 + 5 + 7, 18 = 7 + 11, 19 = 3 + 5 + 11, 20 = 2 + 7 + 11,\\
&21 = 2 + 3 + 5 + 11, 22 = 2 + 7 + 13, 23 = 2 + 3 + 5 + 13, 24 = 11 + 13, 25 = 5 + 7 + 13,\\
&26 = 2 + 11 + 13, 27 = 2 + 5 + 7 + 13, 28 = 3 + 5 + 7 + 13, 29 = 5 + 11 + 13,\\
&30 = 2 + 3 + 5 + 7 + 13, 31 = 2 + 5 + 11 + 13, 32 = 3 + 5 + 11 + 13.
\end{aligned}$$

9. We prove this by induction. It is clear for $n = 1$. Suppose true for n, by Berstrand's postulate there is a prime p with $2n + 2 < p \le 4n + 4$ but p is odd so $2n + 2 < p < 4n + 3$ and we take pairs $\{2n + 2, p - (2n + 2)\}, \{2n + 1, p - (2n + 1)\}, \ldots$. The sum of each pair is the prime p and for the remaining set of integers $\{1, \ldots, p - (2n + 2) - 1\}$ by assumption we can partition these into pairs such that the sum of each pair is a prime.

10. $e = [2, 1, 2, 1, 1, \ldots], \pi = [3, 7, 15, 1, 292, \ldots]$.

11. If $x = [a, a, a, \ldots]$ then $x = a + 1/x$ and so $x^2 - ax - 1 = 0$ which gives $x = \frac{a \pm \sqrt{a^2+4}}{2}$ but x is positive so $x = \frac{a+\sqrt{a^2+4}}{2}$.

12. $\sqrt{3} = [1; 1, 2]$, $\sqrt{7} = [2; 1, 1, 1, 4]$, $\sqrt{13} = [3; 1, 1, 1, 1, 6]$, $\sqrt{19} = [4; 2, 1, 3, 1, 2, 8]$, $\sqrt{46} = [6; 1, 3, 1, 1, 2, 6, 2, 1, 1, 3, 1, 12]$.