Distribution Theory

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1 Introduction and Notations

1.1 Introductions

Why are we interested in distribution theory? In classical manner, we usually understand the function by given the value of that specific function at each point, but what about the functions such that Dirac Delta function. In the light of that, we shall give the following examples and henceforth motivate the application of distribution theory.

Example 1.1. *Find the derivative of $\delta(x-x_0)$. We shall define the Dirac Delta functions by*

$$\int_{-\infty}^{\infty} \delta(x-x_0)f(x)dx = f(x_0).$$

If we use integration by part, which we will assume to be allowed, then we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta(x-x_0)f(x)dx &= \lim_{h\to 0}\int_{-\infty}^{\infty} \frac{1}{h}[\delta(x-x_0+h)-\delta(x-x_0)]f(x)dx \\
&= \lim_{h\to 0}[f(x_0-h)-f(x_0)] \\
&= -f'(x_0) = -\int_{-\infty}^{\infty} \delta(x-x_0)f'(x)dx.
\end{aligned}$$

This suggests that the ordinary rules of integral calculus works for $\delta(x-x_0)$ and distribution theory will make this rigorous.

Example 1.2. *The Fourier Transform and Inverse Fourier Transform for 'nice' functions.*
The Fourier Transform for f is defined by

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x}f(x)dx,$$

and the inverse Fourier Transform is defined by,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\hat{f}(\lambda)d\lambda.$$

So what is $\hat{1}(\lambda)$?. We have

$$\begin{aligned} f(x_0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x_0}\hat{f}(\lambda)d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x_0}\left(\int_{-\infty}^{\infty} e^{-i\lambda x_0} f(x)dx\right)d\lambda \\ &= \int_{-\infty}^{\infty} f(x)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda(x-x_0)}d\lambda\right]dx \\ &= f(x)\delta(x-x_0)dx. \end{aligned}$$

where the last step is straight by definition of Delta function. The above works fine except we need to apply Fubini's theorem to swap the order of integrals and the conditions may not be satisfied. But this suggests that

$$2\pi\delta'(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x}dx = \hat{1}(\lambda).$$

If we differentiate the above, we get

$$2\pi\delta'(\lambda) = \int_{-\infty}^{\infty} xe^{-i\lambda x}dx = \hat{x}(\lambda).$$

Again we will use distribution theory to make this rigorous.

Example 1.3. *Solutions to PDE. Consider the partial differential equation*

$$0 = \frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2}.$$

Then we have

$$\int\int\left(\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2}\right) f(x,t)dxdt = 0.$$

for all $f(x,t) \in C^2(\mathbb{R}^2)$ and $f(x,t) = 0$ when $|x|^2 + |t| > R$ for R large. Apply integration by part, we have

$$\int\int\left(\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2}\right) p(x,t)dxdt = 0.$$

Using distribution theory, we can find the solution to this.

In each of the above examples we introduce a space of 'nice' functions. We extend the rules for these functions into a large class by **duality**. In distribution theory, ordinary functions are replaced by much more general objects (called **generalised functions**). More formally, given a vector space of 'nice' functions V, we define the space of distributions to the set of all continuous linear maps from V to $\mathbb{C}$. Denote this space by V^*. In other words, for $u \in V^*$, we have

$$u(f) := \langle u, f\rangle, \langle u, \alpha f + \beta g\rangle = \alpha\langle u, f\rangle + \beta\langle u, g\rangle.$$

Take for example $V = \mathbb{C}^\infty(\mathbb{R})$, define Dirac Delta function with support at x_0 by

$$\langle \delta_{x_0}, f\rangle = f(x_0),\ \forall f \in V$$

We can easily check that δ_{x_0} is linear. What about continuity.

We need a topology on V or rather a way of saying a sequence $f_n \in V$ converges. What do we mean by $f_n \to f$ in V? We can then say $u : V \to \mathbb{C}$ is continuous if

$$\langle u, f_n\rangle \to \langle u, f\rangle\ \forall f_n \to f \text{ in } V.$$

By using $g_n = f_n - f$, it is equivalent to say that for each $f_n \to 0$ in V, $\langle u, f_n\rangle \to 0$. This is difficult. We need theory of locally convex topological vector space and ultimate Frehet space. In this course, we will assume these exist.

1.2 Notations

We will denote the points in $\mathbb{R}^n$ as $x = (x_1, \dots, x_n)$. The volume elements of $\mathbb{R}^n$ is $dx = dx_1 \cdots dx_n$. We will usually use K to denote compact set in $\mathbb{R}^n$.

Also, we will introduce the multi-index notation. For multi-index α, we mean $\alpha \in \mathbb{Z}_n^+, \alpha = (\alpha_1, \dots, \alpha_n), \alpha_i \in \mathbb{Z}^+$. We write

$$\partial_x^\alpha := \left(\frac{\partial}{\partial x}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x}\right)^{\alpha_n},$$

and

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \alpha! = \alpha_1! \cdots \alpha_n!.$$

We will also assume the following theorem:

Theorem 1.4. [Dominated Convergence Theorem (DCT)] *If $\int_X |f|dx < \infty$ then we write $f \in L^1(X)$. If a sequence $f_n \in L^1(X)$ tends to f almost everywhere, and $|f_n| \leq g$ for all n and some $g \in L^1(X)$, then*

$$f \in L^1(X) \text{ and } \mu(f_n) \to \mu(f),$$

where μ is the measure. In our language, it is just the integration. For shorthand, the theorem concludes

$$\lim_{n\to\infty} \int_X f_n(x)dx = \int_X \lim_{n\to\infty} f_n(x)dx = \int_X f(x)dx.$$

Example 1.5. *For $f \in L^1(\mathbb{R})$, we have by DCT,*

$$\lim_{\lambda \to \lambda_0} \int_{-\infty}^{\infty} e^{i\lambda x} f(x) dx = \int_{-\infty}^{\infty} e^{-i\lambda_0 x} f(x) dx.$$

Finally, for convention, we will always write $\int$ to mean $\int_{-\infty}^{\infty}$ or $\int_{\mathbb{R}^n}$.

2 Distribution and Test Functions

2.1 Basic Properties

Definition 2.1. *We define the support of a function,*

$$supp(f) = cl\{x \in \mathbb{R}^n : f(x) \neq 0\},$$

where cl means the closure.

Definition 2.2. *Let $D(X)$ denote the space of smooth functions with compact support. We say $\psi_m \to 0$ in $D(X)$ if there is some compact subset $K \subset X$, such that $supp(\psi_m) \subset K$ and $\partial^\alpha \psi_m \to 0$* **uniformly** *for each each multi-index α.*

Now for any $\phi, \psi \in D(X)$, we have

$$\int_X \phi \partial^\alpha \psi dx = (-1)^{|\alpha|} \int_X \psi \partial^\alpha \phi dx.$$

were we assume the functions vanish on the boundary as they have compact support. For any $\psi \in D(X)$, we have $\psi(x+h) = \sum_{|\alpha| \leq N} \frac{h^\alpha}{\alpha!} \partial^\alpha \psi(x) + R_N(x,h)$ where $R_N(x,h) \to 0$ in $D(X)$. (See Exercise 1)

Definition 2.3. *If $u : D(X) \to \mathbb{C}$ is linear and for each compact $K \subset X$ there exists some numbers $C, N > 0$ such that*

$$|u(\psi)| := |\langle u, \psi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \psi|.$$

for all $\psi \in D(X)$ with $supp(\psi) \subset K$. Then we say $u \in D'(X)$. If we assume the same N for each compact subset K, then we call the best such N the order of u, written $ord(u) = N$.

Example 2.4. *$\langle \delta_{x_0}, \psi \rangle = \psi(x_0), \delta_{x_0} \in D'(X)$. It is clearly linear and for each ψ and each compact subset K, we have*

$$|\langle \delta_{x_0}, \psi \rangle| = |\psi(x_0)| \leq \sup_k |\psi| = \sum_{|\alpha| \leq 0} |\partial^\alpha \psi|.$$

Hence the order is 0.

Example 2.5. *Define the linear mao $T_M : D(X) \to \mathbb{C}$ by*

$$\langle T_M, \psi \rangle = \sum_{|\alpha| \leq M} \int_X f_\alpha \partial^\alpha \psi dx,$$

where $f_\alpha \in C(X)$. *It is clear linear, now for each* $K \subset X$, *and supp*$(\psi) \subset K$, *we have*

$$\begin{aligned} |\langle T_M, \psi\rangle| = \left|\sum_{|\alpha|\le M}\int_X f_\alpha \partial^\alpha \psi dx\right| &\le \sum_{|\alpha|\le M}\int_K |f_\alpha||\partial^\alpha\psi| dx \\ &\le \sum_{|\alpha|\le M} \sup|\partial^\alpha\psi| \int_K |f_\alpha| dx \end{aligned}$$

where the integral is a constant as K *is compact. Hence, the order of* T_M *is just* M.

Not every distribution has a finite order.

Example 2.6. *Consider the distribution* $u : D(0,1) \to \mathbb{C}$ *by*

$$\langle u, \psi\rangle = \sum_{n=1}^{\infty} \psi^{(n)}\left(\frac{1}{1+n}\right).$$

It is clearly linear. Fix $K \subset (0,1)$. *There is an* $x^* \in K$, *which is minimal, as* K *is compact. Then for* $\psi \in D(0,1)$ *with supp*$(\psi) \subset K$, *we have*

$$\begin{aligned} |\langle u, \psi\rangle| &= \left|\sum_{n=1}^{\infty} \psi^{(n)}\left(\frac{1}{1+n}\right)\right| \\ &\le \sum_{n=1}^{\infty}\left|\psi^{(n)}\left(\frac{1}{1+n}\right)\right| \\ &= \sum_{n=1}^{N}\left|\psi^{(n)}\left(\frac{1}{1+n}\right)\right| \\ &\le \sum_{n\le N} \sup_K \psi^{(n)}\left(\frac{1}{1+n}\right), \end{aligned}$$

where N *is the largest number with* $\frac{1}{1+N} > x^*$. *So it defines a distribution, but note that the constant* N *depends on the choice of the compact subset* K *and so we do not have an order.*

Lemma 2.7. *Let* u *be a linear function from* X *to* $\mathbb{C}$. *Then* $u \in D'(X)$ *if and only if* $\langle u, \psi_m\rangle \to 0$ *for each* $\psi_m \to 0$ *in* $D(X)$.

Proof. For each compact subset K we have C, N such that for all ψ with support in K,

$$|\langle u, \psi\rangle| \le C \sum_{|\alpha|\le N} \sup|\partial^\alpha\psi|.$$

Now let $\psi_m \to 0$ in $D(X)$, it means that we have a compact subset $K \subset X$ such that $\text{supp}(\psi) \in K$ and $\partial^\alpha \psi_m \to 0$ uniformly. Then for this K we have

$$|\langle u, \psi_m \rangle| \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \psi_m| \to 0.$$

Conversely, assume $\langle u, \psi_m \rangle \to 0$ for each $\psi_m \to 0$ in $D(X)$. Suppose u does not satisfy the condition, i.e. there is a compact set K such that for all $C, N > 0$, we have some ψ with $\text{supp}(\psi) \in K$ and

$$|\langle u, \psi \rangle| > C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \psi|.$$

In particular, we set C, N to be m and hence get a sequence ψ_m for which the above hold. Let $\phi_m = \frac{\psi_m}{|\langle u, \psi_m \rangle|}$. Then we have

$$1 > m \sum_{|\alpha| \leq m} \sup_K |\partial^\alpha \phi_m|.$$

and so $\sup_K |\partial^\alpha \phi_m| < \frac{1}{m}$ for all $m \geq |\alpha|$. Hence, we see that $\phi_m \to 0$ in $D(X)$ by definition. However, $\langle u, \phi_m \rangle = 1$ for all m, which is a contradiction. □

2.2 Limits in Distributions

Often one has a sequence of distributions $u_k \in D'(X)$. It is natural to ask if the sequence converges.

Definition 2.8. *We say $u_k \to u$ in $D'(X)$ if $\langle u_k, \psi \rangle \to \langle u, \psi \rangle$ for all $\psi \in D(X)$.*

We have a hard but useful theorem (which we are not going to prove here)

Theorem 2.9. *If $u_k \in D'(X)$, and $\langle u, \psi \rangle = \lim_{k \to \infty} \langle u_k, \psi \rangle$ where the limit exists for all $\psi \in D(X)$. Then $u \in D'(X)$.*

The limit in $D'(X)$ might be a bit strange:

Example 2.10. $u_k(x) = \cos(kx)$. *Then*

$$\langle u_k, \psi \rangle = \int \cos(kx)\psi(x)dx = -\frac{1}{k} \int \sin(kx)\psi(x)dx,$$

which tends to 0 *as $k \to \infty$. Thus, $u_k \to 0$ in $D'(X)$.*

Example 2.11. *$\psi \in D(\mathbb{R})$ with $\int \psi(x)dx = 1$. Define the sequence $\psi_k(x) = k\psi(kx)$. Then we have*

$$\begin{aligned}\langle \psi_k(x), \phi \rangle = \int \psi_k(x)\phi(x)dx &= \int k\psi(kx)\phi(x)dx \\ &= \int \psi(y)\phi\left(\frac{y}{k}\right)dy \to \phi(0)\int \phi(y)dy = \phi(0)\end{aligned}$$

where in the second line we used the substitution $y = kx$ and then apply dominated convergence theorem. Thus, $\langle \psi_k, \phi \rangle \to \phi(0) = \langle \delta_0, \phi \rangle$ and so we conclude that $\psi_k \to \delta_0$. Similarly, one can show the function $\psi_\epsilon(x) \to \delta_0$ as $\epsilon \to 0$ where $\psi_\epsilon(x) = \frac{1}{\epsilon}\psi(\frac{x}{\epsilon})$.

2.3 Basic Operations on Distributions

For any $u \in C^\infty(X)$, we can always define the distribution $\partial^\alpha u$ as

$$\langle \partial^\alpha u, \psi \rangle = \int_X (\partial^\alpha u)\psi dx.$$

Integrating by part, we have

$$\langle \partial^\alpha u, \psi \rangle = \langle u, (-1)^{|\alpha|}\partial^\alpha \psi \rangle.$$

This suggests that:

Definition 2.12. *For any $f \in C^\infty(X)$ and any multi-index α, we define*

$$\langle fu, \psi \rangle = \langle u, f\psi \rangle, \langle \partial^\alpha u, \psi \rangle = (-1)^{|\alpha|}\langle u, \partial^\alpha \psi \rangle.$$

We call $\partial^\alpha u$ the distribution derivative of u.

Example 2.13.

$$\langle \partial^\alpha \delta_{x_0}, u \rangle = (-1)^{|\alpha|}\langle \delta_{x_0}, \partial^\alpha \psi \rangle = (-1)^{|\alpha|}\partial^\alpha \psi(x_0).$$

Example 2.14. *Define the heviside step function*

$$H(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

In the sense of distribution, we have

$$\langle H, \psi \rangle = \int_0^\infty \psi(x)dx.$$

Then by definition, H' is defined by

$$\langle H', \psi \rangle = (-1)\langle H, \psi' \rangle = -\int_0^\infty \psi'(x)dx = \psi(0).$$

which is the same as $\langle \delta_0, \psi \rangle$. Hence $H' = \delta_0$.

Lemma 2.15. *If $u \in D'(\mathbb{R})$ and $u' = 0$ in $D'(\mathbb{R})$. Then u is constant.*

Proof. Let $\theta(x) \in D(\mathbb{R})$ such that $\int \theta(x)dx = 1$. Then for any $\psi(x) \in D(\mathbb{R})$,we have

$$\psi(x) = \{\psi(x) - \langle 1, \psi\rangle\theta(x)\} + \langle 1, \psi\rangle\theta(x) = \psi_A(x) + \psi_B(x).$$

Further,

$$\langle 1, \psi_A(x)\rangle = \{\langle 1, \psi(x)\rangle\}\{1 - \langle 1, \theta(x)\rangle\} = 0.$$

Now define $\phi(x) = \int_{-\infty}^{x} \psi_A(y)dy$. Since $\langle 1, \psi_A(x)\rangle = 0$, and so $\int_{-\infty}^{\infty} \psi_A(y)dy = 0$. So for x large, $\phi(x) = 0$. Also, for x small, $\phi(x) = 0$ because $\psi_A(y) = 0$ for y small enough. Hence, $\phi(x) \in D'(\mathbb{R})$.

Now since $\phi'(x) = \psi_A(x)$,

$$\langle u, \psi\rangle = \langle u, \psi_A\rangle + \langle u, \psi_B\rangle = \langle u, \phi'(x)\rangle + \langle u, \psi_B\rangle.$$

The first term is by definition -$\langle u', \phi(x)\rangle = 0$ by assumption of u'. The second term is

$$\langle u, \psi_B\rangle = \langle 1, \psi\rangle\langle u, \theta\rangle = C\langle 1, \psi\rangle = \langle C, \psi\rangle,$$

where $C = \langle u, \theta\rangle$ is a constant. □

2.4 Reflection and Translation

Definition 2.16. *If $u \in C^\infty(\mathbb{R}^n)$, then we can define*

$$\check{u}(x) = u(-x), (\tau_h u)(x) = u(x-h).$$

Treating $u \in D'(X)$, we have

$$\langle \check{u}, \psi\rangle = \langle u, \check{\psi}\rangle, \langle \tau_h u, \psi\rangle = \langle u, \tau_{-h}\psi\rangle.$$

Lemma 2.17. *For $u \in D'(\mathbb{R})$, define*

$$V_h = \frac{\tau_{-h}u - u}{|h|}.$$

Then $\lim_{h\to 0} V_h = \underline{n} \cdot \partial u$, where $\underline{n} = \lim_{h\to 0} \frac{h}{|h|}$ is a unit vector.

Proof. Let $U_h = \tau_{-h}u - u$, then we have

$$\langle U_h, \psi\rangle = \langle u, \tau_h\psi\rangle - \langle u, \psi\rangle = \langle u, \tau_h\psi - \psi\rangle.$$

Now, by Taylor expansion we have

$$\tau_h\psi(x) - \psi(x) = \psi(x-h) - \psi(x) = -\sum_{i=1}^{n} h_i \frac{\partial\psi}{\partial x_i} + R.$$

It is easy to show (See Exercise 1) that $\partial^\alpha R = o(|h|)$ uniformly in x and $\mathrm{supp}(\mathbb{R})$ is contained in some fixed compact set for $|h| < 1$. So we conclude that $\frac{R}{|h|} \to 0$ as $h \to 0$.

Since $V_h = \frac{U_h}{|h|}$, we have

$$\begin{aligned} \langle V_h, \psi \rangle &= \left\langle u, -\sum_{i=1}^n \frac{h_i}{|h|}\frac{\partial \psi}{\partial x_i} + \frac{R}{|h|} \right\rangle \\ &\to \langle u, -\underline{n} \cdot \partial \psi \rangle \\ &= \langle \underline{n} \cdot \partial u, \psi \rangle. \end{aligned}$$

Hence, $V_h \to \underline{n} \cdot \partial u$. □

2.5 Convolution

For $u, \psi \in D(\mathbb{R})$, the convolution of two functions is defined by

$$(u * \psi)(x) = \int u(x-y)\psi(y)dy = \int u(y)\psi(x-y)dy = (\psi * u)(x).$$

We can also write it as $\langle u, \tau_{-x}\check{\psi} \rangle$. This suggests:

Definition 2.18. *For any $\psi \in D(\mathbb{R}^n)$, and $u \in D'(\mathbb{R}^n)$, define the convolution*

$$(u * \psi)(x) = \langle u, \tau_{-x}\check{\psi} \rangle, \text{ where } (\tau_{-x}\check{\psi})(y) = \psi(x-y),$$

and $u = u(y)$ acts on y. To be precise, sometimes we will write $u = u(y)$ to mean u acts on the variable y.

Lemma 2.19. *Let $\Phi_x(y) = \phi(x,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $\Phi(\cdot, y) = 0$ for $y \notin K$, where $K \subset \mathbb{R}^n$ is compact. Then*

$$\partial_x^\alpha \langle u(y), \Phi_x \rangle = \langle u, \partial_x^\alpha \Phi_x \rangle.$$

Proof. By Taylor expansion,

$$\Phi_{x+h}(y) - \Phi_x(y) = \sum_{i=1}^n h_i \frac{\partial \Phi}{\partial x_i}(x,y) + R,$$

and $R = o(|h|)$ in $D(\mathbb{R}^n)$. So we have

$$\langle u, \Phi_{x+h} - \Phi_x \rangle = \sum_{i=1}^n h_i \langle u, \frac{\partial \Phi}{\partial x_i}(x,y) \rangle + o(|h|).$$

Taking limit $h \to 0$, we have

$$\frac{\partial}{\partial x_i}\langle u, \Phi_x \rangle = \langle u, \frac{\partial \Phi}{\partial x_i}(x, y) \rangle.$$

□

Corollary 2.20. *If $u \in D'(\mathbb{R})$ and $\psi \in D(\mathbb{R}^n)$ then $u * \psi \in C^\infty(\mathbb{R}^n)$ and $\partial^\alpha(u * \psi) = u * (\partial^\alpha \psi)$.*

Proof. By definition

$$(u * \psi)(x) = \langle u, \tau_{-x}\check{\psi} \rangle = \langle u, \phi \rangle,$$

where $\phi(x, y) = \psi(x - y)$. By previous lemma, we have

$$\partial_x^\alpha \langle u, \tau_{-x}\check{\psi} \rangle = \langle u, \partial_x^\alpha \tau_{-x}\check{\psi} \rangle.$$

Also,

$$u * (\partial^\alpha \psi)(x) = \langle u, \tau_{-x}\check{\partial_y^\alpha} \psi \rangle.$$

Finally, just observe that

$$\partial_x^\alpha \psi(x - y) = \tau_{-x}\partial_y^\alpha(\psi)|_{-y}.$$

□

2.6 Density

Lemma 2.21. *For $\phi, \psi \in D(\mathbb{R}^n)$ and $u \in D'(\mathbb{R}^n)$, we have*

$$(u * \phi) * \psi = u * (\phi * \psi).$$

Proof. By definition, we have

$$\begin{aligned}
[(u * \phi) * \psi](x) &= \int (u * \phi)(x - y)\psi(y)dy \\
&= \int \langle u, \tau_{-x+y}\check{\phi} \rangle \psi(y)dy \\
&= \int \langle u(z), \phi(x - y - z) \rangle \psi(y)dy \\
&= \lim_{h \to 0} \sum_{m \in \mathbb{Z}^n} \langle u(z), \phi(x - hm - z)\psi(hm)h^n \rangle \\
&= \langle u(z), \lim_{h \to 0} \sum_{m \in \mathbb{Z}^n} \psi(hm)\phi(x - hm - z)h^n \rangle \\
&= \langle u(z), \int \psi(y)\phi(x - y - z)dy \rangle \\
&= \langle u(z), (\phi * \psi)(x - z) \rangle \\
&= [u * (\phi * \psi)](x).
\end{aligned}$$

where in the fourth line we used the definition of a Riemann integral in terms of an infinite sum (which then allows us to swap the order of the sum and the action of u because ϕ, ψ have compact support so the sum is finite). $\square$

Theorem 2.22. *$D(\mathbb{R}^n)$ is dense in $D'(\mathbb{R}^n)$.*

Proof. Let $\phi \in D(\mathbb{R}^n)$ such that $\int \phi(x)dx = 1$. Let $\phi_k(x) = k^n\phi(kx)$ so by Example 1.11 we have seen that $\phi_k \to \delta_0$ in $D'(\mathbb{R}^n)$. Also we shall introduce the cutoff function $\chi \in D(\mathbb{R}^n)$ by

$$\chi(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 2. \end{cases}$$

and we will write $\chi_k(x) = \chi\left(\frac{x}{k}\right)$. Let $u \in D'(\mathbb{R}^n)$, consider

$$\psi_k(x) = \chi_k(x)(u * \phi_k)(x).$$

We have shown $u * \phi_k$ is smooth so ψ_k is smooth. Also since χ_k has compact support, so does ψ_k. So $\psi_k \in D(\mathbb{R}^n)$.

For any $\theta \in D(\mathbb{R}^n)$, we have

$$\langle \psi_k, \theta\rangle = \langle u * \phi_k, \chi_k\theta\rangle.$$

Observe that $(u * \check{\psi})(0) = \langle u, \tau_0\check{\check{\psi}}\rangle = \langle u, \psi\rangle$. So

$$\langle \psi_k, \theta\rangle = ((u * \phi_k) * \check{\chi_k\theta})(0).$$

By previous lemma, it is the same as $u * [\phi_k * (\check{\chi_k\theta})](0)$.

We have

$$\begin{aligned}
[\phi_k * (\check{\chi_k\theta})](x) &= \int \phi_k(x-y)\chi_k(-y)\theta(-y)dy \\
&= \int k^n\phi(kx-ky)\chi\left(\frac{-y}{x}\right)\theta(-y)dy \\
&= \int \phi(y')\chi\left(\frac{y'}{k^2} - \frac{x}{k}\right)\theta\left(\frac{y'}{k} - x\right)dy' \\
&= \int \phi(y')\left[\chi\left(\frac{y'}{k^2} - \frac{x}{k}\right)\theta\left(\frac{y'}{k} - x\right)dy - \theta(-x)\right]dy' \\
&+ \int \theta(-x)\phi(y')dy' \\
&= R_k(-x) + \theta(-x).
\end{aligned}$$

where we used the substitution $y' = kx - ky$. Setting $x = 0$, we have

$$\langle \psi_k, \theta\rangle = (u * \check{R}_k)(0) + (u * \check{\theta})(0) = \langle u, R_k\rangle + \langle u, \theta\rangle.$$

It is easy to show that $R_k \to 0$ as $k \to \infty$ by using DCT, and so

$$\langle \psi_k, \theta \rangle \to \langle u, \theta \rangle.$$

The result follows because θ is arbitrary. $\square$

3 Distributions With Compact Support

3.1 Basic Properties

Definition 3.1. *We say that a distribution $u \in D'(X)$ vanishes on $Y \subset X$ if $\langle u, \psi \rangle = 0$ for all $\psi \in D(Y)$. Then we define supp(u) as the complement of the largest open set on which u vanishes.*

Example 3.2. *For $x_0 \in X$, supp$(\delta_{x_0}) = \{x_0\}$.*

Definition 3.3. *The space $\epsilon(X)$ is the space consisting of smooth functions from X to $\mathbb{C}$. We say a sequence $\psi_k(x) \to 0$ in $\epsilon(X)$ if $\partial^\alpha \psi_k \to 0$ uniformly on each compact subset of X, for each multi-index α.*

Definition 3.4. *A linear map $u : \epsilon(X) \to \mathbb{C}$ is an element of $\epsilon'(X)$ if there exists a compact set $K \subset X$ and numbers $C, N > 0$ such that*

$$|\langle u, \psi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \psi|,$$

for all $\psi \in \epsilon(X)$.

Lemma 3.5. *$u \in \epsilon'(X)$ if and only if $\lim_{k \to \infty} \langle u, \psi_k \rangle = 0$ whenever $\psi_k \to 0$ in $\epsilon(X)$.*

The proof is very similar to the one for $D'(X)$.

Lemma 3.6. *If $u \in \epsilon'(X)$, then u restricted to $D(X)$ gives a distribution in $D'(X)$ with compact support and finite order. Conversely, each $u \in D'(X)$, with compact support has a unique extension to $\hat{u}$ in $\epsilon'(X)$. So we can understand $u \in \epsilon'(X)$ as a distribution with compact support.*

Proof. Notice that $D(X) \subset \epsilon(X)$ so the restriction is well-defined. Let $u \in \epsilon'(X)$, so u has a compact support, say $\hat{K}$. Then for each $\psi \in D(X)$, with support in a compact subset K, we have $C, N > 0$ such that

$$|\langle u, \psi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_{\hat{K} \cap K} |\partial^\alpha \psi| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \psi|.$$

Hence it is a distribution in $D'(X)$ and it is true for each K so it has finite order, N.

Conversely, let $u \in D'(X)$ with compact support K. Let $\rho \in D(X)$ such that $\rho = 1$ on a neighbourhood of K. Thus, for each $\psi \in \epsilon(X)$, we extend u by

$$\langle \hat{u}, \psi \rangle = \langle u, \rho\psi \rangle,$$

on noting that $\rho\psi$ has compact support. Then

$$|\langle \hat{u}, \psi\rangle| = |\langle u, \rho\psi\rangle| \leq C \sum_{|\alpha|\leq N} \sup_K |\partial^\alpha(\rho\psi)|,$$

because u has support K, and we know $\rho = 1$ on K, hence the condition is satisfied.

If we have two extensions u_1, u_2. Then u_1, u_2, u agrees on $D(X)$ (writing $\psi = \rho\psi + \psi(1-\rho)$) and so for any ψ we have

$$\langle u_1, \psi\rangle = \langle u_1, \rho\psi\rangle = \langle u, \rho\psi\rangle = \langle u_2, \rho\psi\rangle = \langle u_2, \psi\rangle.$$

□

So now we can understand $u \in \epsilon'(X)$ as distribution with compact support.

3.2 Convolution

Take $u \in \epsilon'(\mathbb{R}^n)$ and $\psi \in \epsilon(\mathbb{R}^n)$. Then $(u * \psi)(x) = \langle u(y), \psi(x-y)\rangle$ and so it is 0 unless $x - y \in \text{supp}(\psi)$ for some $y \in \text{supp}(u)$. This is because, if $\phi(y) = \psi(x-y)$, then $\langle u, \phi(y)\rangle \neq 0$ when $\phi(y) \neq 0$ for some $y \in \text{supp}(u)$. Thus, $x \in \text{supp}(\psi)+\text{supp}(u)$.

Definition 3.7. *Given $u_1, u_2 \in D'(\mathbb{R}^n)$ and at least one of u_1, u_2 has compact support. Then we can define $u_1 * u_2$ to be the unique $u \in D'(\mathbb{R}^n)$ such that*

$$u_1 * (u_2 * \psi) = u * \psi, \ \forall \psi \in D(\mathbb{R}^n).$$

Lemma 3.8. *The definition above is well-defined and it is indeed unique.*

Proof. If u_1 has compact support. We know $u_2 * \psi$ is smooth and so is in $\epsilon(\mathbb{R}^n)$. If u_2 has compact support, then $u_2 * \psi \in D(\mathbb{R}^n)$ and so it is again well-defined.

Now for uniqueness it suffices to show that if $u * \psi = 0$ for all $\psi \in D(\mathbb{R}^n)$ then $u = 0$. But $\langle u, \psi\rangle = (u * \check{\psi})(0) = 0$ for all $\psi \in D(\mathbb{R}^n)$ and so $u = 0$. □

Lemma 3.9. *Let $u_1, u_2 \in D'(\mathbb{R})$ with at least one of supp(u_i) compact. Then $u_1 * u_2 = u_2 * u_1$.*

Proof. For any $\psi, \phi \in D(\mathbb{R})$, we have

$$\begin{aligned}(u_1 * u_2) * (\psi * \phi) &= u_1 * (u_2 * (\psi * \phi)) \\ &= u_1 * ((u_2 * \psi) * \phi) \\ &= u_1 * (\phi * (u_2 * \psi)) \\ &= (u_1 * \phi) * (u_2 * \psi)\end{aligned}$$

where in the second line and last line we used Lemma 2.21. Then interchange u_1 and u_2, we have

$$(u_2 * u_1) * (\phi * \psi) = (u_2 * \psi) * (u_1 * \phi) = (u_1 * \phi) * (u_2 * \psi).$$

Thus, we have

$$(u_1 * u_2) * (\psi * \phi) = (u_2 * u_1) * (\psi * \phi).$$

Hence, $u * (\psi * \phi) = (u * \psi) * \phi = 0$, where $u = u_1 * u_2 - u_2 * u_1$. Now

$$[(u * \psi) * \phi](x) = \langle u * \psi, \tau_{-x}\check{\phi} \rangle = 0, \ \forall \psi, \phi, x.$$

Hence, $u * \psi = 0$ for all $\psi \in D(\mathbb{R})$. Then,

$$(u * \psi)(x) = \langle u, \tau_{-x}\check{\psi} \rangle = 0, \ \forall \psi, x.$$

Therefore, we conclude that $u = 0$. □

Remark 3.10. *From the proof above we can also see that, $u * \psi = 0$ for all ψ then $u = 0$.*

Lemma 3.11. *Let $u \in D'(\mathbb{R}^n)$, and $v \in \epsilon'(\mathbb{R}^n)$. Then if $\psi_k \in D(\mathbb{R}^n)$ is such that $\psi_k \to u$ in $D'(\mathbb{R}^n)$. Then $\psi_k * v \to u * v$.*

Proof. For any $\phi \in D(\mathbb{R}^n)$, and since ψ is automatically in $\epsilon'(\mathbb{R}^n)$, then

$$\begin{aligned}
\lim_{k\to\infty} [(\psi_k * v) * \phi](x) &= \lim_{k\to\infty} [\psi_k * (v * \phi)](x) \\
&= \lim_{k\to\infty} \langle \psi_k, \tau_{-x}(v * \phi)^{\vee} \rangle \\
&= \langle u, \tau_{-x}(v * \phi)^{\vee} \rangle \\
&= [u * (v * \phi)](x) \\
&= [(u * v) * \phi](x),
\end{aligned}$$

Hence, $\phi_k * v \to u * v$. □

Example 3.12. *Consider*

$$\delta_0 * \phi(x) = \langle \delta_0, \tau_{-x}\check{\phi} \rangle = \phi(x).$$

Hence,

$$(\delta_0 * u) * \phi = (u * \delta_0) * \phi = u * (\delta_0 * \phi) = u * \phi.$$

4 Tempered Distributions and Fourier Transform

4.1 Functions and Rapid Decay

Definition 4.1. *The Schwarz class of test functions consists of smooth functions from $\mathbb{R}^n$ to $\mathbb{C}$ such that*

$$\|\psi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \psi| < \infty.$$

The space of such functions is denoted by $\mathcal{S}(\mathbb{R}^n)$. We say $\psi_k \in \mathcal{S}(\mathbb{R}^n)$ tends to 0 if $\|\psi\|_{\alpha,\beta} \to 0$ for each multi-index α, β. For example, $\psi(x) = e^{-|x|^2}$ is a Schwarz function.

Definition 4.2. *A linear map $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is called a tempered distribution if there exist constants $C, N > 0$ such that*

$$|\langle u, \psi\rangle| \leq C \sum_{|\alpha|,|\beta|\leq N} \|\psi\|_{\alpha,\beta}.$$

The space of all such linear maps is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

Remark 4.3. *(i) Note that $D(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \epsilon(\mathbb{R}^n)$. The inclusions are continuous, i.e. for each sequence $\psi_k \in D(\mathbb{R}^n)$ we have if $\psi_k \to 0$ in $D(\mathbb{R}^n)$ then $\psi_k \to 0$ in $\mathcal{S}(\mathbb{R}^n)$ and so on.*

(ii) It is also easy to check by definition that $\epsilon'(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset D'(\mathbb{R})^n$. Also the inclusions are continuous. (We can think of $D'(\mathbb{R}^n)$ as the dual of $D(\mathbb{R}^n)$)

4.2 The Fourier Transform

Definition 4.4. *For $f \in L^1(\mathbb{R})$, we define the Fourier Transform by*

$$\hat{f}(\lambda) = \int e^{-i\lambda x} f(x)dx.$$

Remark 4.5. $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ *because for $\psi \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\begin{aligned}
\int |\psi(x)|dx &= \int (1+|x|)^{-N}(1+|x|)^N|\psi(x)|dx \\
&\leq \sup_{x\in\mathbb{R}^n} \left\{(1+|x|)^N|\psi(x)|\right\} \int (1+|x|)^{-N}dx \\
&\leq C \sum_{|\alpha|\leq N} \|\psi\|_{\alpha,0},
\end{aligned}$$

where we pick N large so that the integral converges.

Lemma 4.6. *If $f \in L^1(\mathbb{R}^n)$ then $\hat{f}$ is continuous.*

Proof. Take a sequence $\lambda_k \in \mathbb{R}^n$ such that $\lambda_k \to \lambda$. Then

$$\hat{f}(\lambda_k) = \int e^{-i\lambda_k x} f(x)dx \to \int e^{-i\lambda x} f(x)dx = \hat{f}(\lambda),$$

using DCT. So $\hat{f}$ is continuous. □

We introduce the notation $D = -i\partial$.

Lemma 4.7. *For $\psi \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$(D^\alpha \psi)^\wedge(\lambda) = \lambda^\alpha \hat{\psi}(\lambda), (x^\beta \psi)^\wedge(\lambda) = (-D)^\beta \hat{\psi}(\lambda).$$

Proof.

$$\begin{aligned}(D^\alpha \psi)^\wedge(\lambda) &= \int e^{-i\lambda x} D^\alpha \psi(x)dx \\ &= (-1)^{|\alpha|} \int D_x^\alpha e^{-i\lambda x} \psi(x)dx \\ &= (-1)^{2|\alpha|} \int \lambda^\alpha e^{-i\lambda x} \psi(x)dx \\ &= \lambda^\alpha \hat{\psi}(\lambda).\end{aligned}$$

$$\begin{aligned}(-D)^\beta \hat{\psi}(\lambda) &= \int (-D)_\lambda^\beta e^{-i\lambda x} \psi(x)dx \\ &= \int x^\beta e^{-i\lambda x} \psi(x)dx \\ &= (x^\beta \psi)^\wedge(\lambda).\end{aligned}$$

□

Theorem 4.8. *The Fourier Transform is an continuous automorphism of $\mathcal{S}(\mathbb{R}^n)$.*

Proof. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned}
\|\hat{\psi}(\lambda)\|_{\alpha,\beta} &= |\lambda^\alpha \partial^\beta \hat{\psi}(\lambda)| = |\lambda^\alpha (-D)^\beta \hat{\psi}(\lambda)| \\
&= |\lambda^\alpha (x^\beta \psi)^\wedge(\lambda)| \\
&= |(D^\alpha (x^\beta \psi))^\wedge(\lambda)| \\
&= \left| \int e^{-i\lambda x} D^\alpha (x^\beta \psi) dx \right| \\
&\leq \int |D^\alpha (x^\beta \psi)|(1+|x|)^N (1+|x|)^{-N} dx \\
&\leq \sup_x \left\{ |D^\alpha (x^\beta \psi)|(1+|x|)^N \right\} \int (1+|x|)^{-N} \\
&\leq C \sum_{|\alpha'|,|\beta'| \leq N'} \|\psi\|_{\alpha',\beta'},
\end{aligned}$$

where in the last line we just expand $D^\alpha(x^\beta \psi)$.

Now from the above estimate, we can see that $\hat{\psi}$ is also in $\mathcal{S}(\mathbb{R}^n)$ and if $\psi_k \to 0$ in $S(\mathbb{R}^n)$ then so is $\hat{\psi}$. So the Fourier Transform is a continuous map. Consider

$$\psi(x) = \frac{1}{(2\pi)^n} \int e^{i\lambda x} \hat{\psi}(\lambda) d\lambda.$$

We claim this is the inverse. We shall use Fubini's theorem, which says that for a double integral, if each integral converges absolutely, then we can interchange the order of integration. To prove this, we have, applying DCT,

$$\begin{aligned}
\int e^{i\lambda x}\hat{\psi}(\lambda)d\lambda &= \lim_{\epsilon\to 0}\int e^{i\lambda x-\epsilon|\lambda|^2}\hat{\psi}(\lambda)d\lambda \\
&= \lim_{\epsilon\to 0}\int e^{i\lambda x-\epsilon|\lambda|^2}\left(\int \psi(y)e^{-i\lambda y}dy\right)d\lambda \\
&= \lim_{\epsilon\to 0}\int \psi(y)\left(\int e^{i\lambda(x-y)-\epsilon|\lambda|^2}d\lambda\right)dy \\
&= \lim_{\epsilon\to 0}\int \psi(y)\left(\prod_{j=1}^{n}\int e^{i\lambda_j(x_j-y_j)-\epsilon\lambda_j^2}d\lambda_j\right)dy \\
&=^* \lim_{\epsilon\to 0}\int \psi(y)\left(\prod_{j=1}^{n}\left(\frac{\pi}{\epsilon}\right)^{\frac{1}{2}}e^{-\frac{(x_j-y_j)^2}{4\epsilon}}\right)dy \\
&= \lim_{\epsilon\to 0}\int \psi(y)\left(\frac{\pi}{\epsilon}\right)^{\frac{n}{2}}e^{-\frac{|x-y|^2}{4\epsilon}}dy \\
&= \lim_{\epsilon\to 0}\int \psi(x-2\sqrt{\epsilon}y')\left(\frac{\pi}{\epsilon}\right)^{\frac{n}{2}}e^{-|y'|^2}(2\sqrt{\epsilon})^n dy' \\
&= 2^n\pi^{\frac{n}{2}}\int \psi(x)e^{-|y|^2}dy' \\
&= 2^n\pi^n\psi(x),
\end{aligned}$$

where we will justify the step (*). In fact, we need to check that

$$\int e^{i\lambda(x-y)-\epsilon\lambda^2}d\lambda = \left(\frac{\pi}{\epsilon}\right)^{\frac{1}{2}}e^{-\frac{(x-y)^2}{4\epsilon}}.$$

But this can be done by setting $\alpha = x-y$ and so

$$\begin{aligned}
\int e^{-\epsilon[\lambda-\frac{i\alpha}{2\epsilon}]^2-\frac{\alpha^2}{4\epsilon}}d\lambda &= e^{-\frac{\alpha^2}{4\epsilon}}\int e^{-\epsilon[\lambda-\frac{i\alpha}{2\epsilon}]^2}d\lambda \\
&= e^{-\frac{\alpha^2}{4\epsilon}}\int_\gamma e^{-\epsilon\mu^2}d\mu \\
&= e^{-\frac{\alpha^2}{4\epsilon}}\int_{\mathbb{R}} e^{-\epsilon\mu^2}d\mu \\
&= \left(\frac{\pi}{\epsilon}\right)^{\frac{1}{2}}e^{-\frac{(x-y)^2}{4\epsilon}},
\end{aligned}$$

where we set $\mu = \lambda - \frac{i\alpha}{2\epsilon}$ and used Cauchy theorem in the third line.

Now by the same method we can show that if

$$\phi(x) = \frac{1}{(2\pi)^n}\int e^{i\lambda x}\hat{\lambda}(\lambda)d\lambda,$$

then

$$\hat{\psi}(\lambda) = \int e^{-i\lambda x}\psi(x)dx.$$

Hence the Fourier Transform is an isomorphism. □

Lemma 4.9. *For $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\int \psi(x)\hat{\phi}(x)dx = \int \hat{\psi}(x)\phi(x)dx.$$

Proof. The proof is straightforward by using Fubini's theorem to interchange the order of integration. □

Corollary 4.10. [Parsval's identity]

$$\int |\psi(x)|^2 dx = \frac{1}{2\pi}\int |\hat{\psi}(\lambda)|^2 d\lambda.$$

Proof. Let $\phi(x) = \bar{\hat{\psi}}(x)$ as in the previous lemma we have

$$\int \psi(x)\hat{\bar{\hat{\psi}}}(x)dx = \int |\hat{\psi}(x)|^2 dx.$$

Let $u(x) = \bar{\hat{\psi}}(x) = \int e^{i\lambda x}\bar{\psi}(\lambda)d\lambda$. We want $\hat{u}$. But Fourier Transform is an isomorphism, and let

$$\psi_1(x) = \frac{1}{2\pi}\int e^{i\lambda x}\bar{\psi}(\lambda)d\lambda = \frac{1}{2\pi}u(x).$$

Then we know $\hat{\psi}_1(x) = \bar{\psi}(x)$. Since $\hat{u} = 2\pi\psi_1(x)$, so we have

$$\hat{u} = 2\pi\hat{\psi}_1(x) = 2\pi\bar{\psi}(x).$$

Thus, use previous expression, we conclude that

$$2\pi\int |\psi(x)|^2 = \int |\hat{\psi}(x)|^2 dx.$$

□

Definition 4.11. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$. We define $\hat{u}$ by*

$$\langle \hat{u}, \psi\rangle = \langle u, \hat{\psi}\rangle,$$

for all $\psi \in \mathcal{S}(\mathbb{R}^n)$. It is easy to check that $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$ because

$$|\langle \hat{u}, \psi\rangle| = |\langle u, \hat{\psi}\rangle| \le C\sum_{|\alpha|,\beta|\le N}\|\hat{\psi}\|_{\alpha,\beta} \le C'\sum_{|\alpha'|,|\beta'|\le N'}\|\psi\|_{\alpha',\beta'},$$

by previous estimate.

Example 4.12.

$$\langle \hat{\delta}_0, \psi \rangle = \langle \delta_0, \hat{\psi} \rangle = \hat{\psi}(0) = \langle 1, \psi \rangle.$$

So $\hat{\delta}_0 = 1$.

Theorem 4.13. *The Fourier Transform is a continuous automorphism on* $\mathcal{S}'(\mathbb{R}^n)$.

Proof. For continuity, let $u_k \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$. Then

$$\langle \hat{u}_k, \psi \rangle = \langle u_k, \hat{\psi} \rangle \to 0,$$

for all $\psi \in \mathcal{S}(\mathbb{R}^n)$. We claim that $\check{u} = (2\pi)^{-n}(\hat{u})^\wedge$. Indeed,

$$\psi(-x) = \frac{1}{(2\pi)^n} \int e^{-i\lambda x} \hat{\psi}(\lambda) d\lambda = \frac{1}{(2\pi)^n} (\hat{\psi})^\wedge.$$

So

$$\langle \check{u}, \psi \rangle = \langle u, \check{\psi} \rangle = \langle (2\pi)^{-n}(\hat{u})^\wedge, \psi \rangle.$$

Therefore, the Fourier Transform is an isomorphism. □

4.3 Sobolev Space

Definition 4.14. *The Sobolev space consists of functions defined in terms of* $\mathcal{S}'(\mathbb{R}^n)$. *For* $s \in \mathbb{R}$, *we define* $H^s(\mathbb{R}^n)$ *as the set of all* $u \in \mathcal{S}'(\mathbb{R}^n)$ *such that* $\hat{u}(\lambda)$ *is a smooth function and*

$$\|u\|_{H^s} = \left[\int |\hat{u}(\lambda)|^2 (1 + |\lambda|^2)^s d\lambda \right]^{\frac{1}{2}} < \infty.$$

For convention, let $\langle \lambda \rangle = (1 + |\lambda|^2)^{\frac{1}{2}}$.

Lemma 4.15. *If* $s > \frac{n}{2}$ *and* $u \in H^s(\mathbb{R}^n)$, *then* u *is a continuous function.*

Proof. We firstly show that $\hat{u} \in L^1(\mathbb{R}^n)$. We have, by Cauchy-Schwarz inequality,

$$\int |\hat{u}(\lambda)| d\lambda = \int \langle \lambda \rangle^{-s} \langle \lambda \rangle^{s} |\hat{u}(\lambda)| d\lambda \leq \left(\int \langle \lambda \rangle^{-2s} d\lambda \right)^{\frac{1}{2}} \left(\int |\hat{u}(\lambda)|^2 \langle \lambda \rangle^{2s} d\lambda \right)^{\frac{1}{2}}.$$

Now $-2s < -n$ by assumption, so by polar coordinate, that $d\lambda = r^{n-1} dr d\sigma$, where $d\sigma$ is the surface element on the n-dimensional sphere, so

$$\int \langle \lambda \rangle^{-2s} d\lambda = \int_{S^{n-1}} d\sigma \int \frac{r^{n-1}}{(1 + r^2)^s} dr,$$

which converges and so is a constant. The second term also converges as $u \in H^s(\mathbb{R}^n)$. So $\int |\hat{u}(\lambda)| d\lambda$ is finite and so $\hat{u} \in L^1(\mathbb{R}^n)$.

Now, consider

$$\begin{aligned}
\langle u, \hat{\psi} \rangle &= \langle \hat{u}, \psi \rangle = \int \hat{u}(\lambda)\psi(\lambda) d\lambda \\
&= \int \hat{u}(\lambda) \left[\frac{1}{(2\pi)^n} \int e^{i\lambda x} \hat{\psi}(x) dx \right] d\lambda \\
&= \int \hat{\psi}(x) \left[\frac{1}{(2\pi)^n} \int e^{i\lambda x} \hat{u}(\lambda) d\lambda \right] dx \\
&= \langle u, \hat{\psi} \rangle,
\end{aligned}$$

using Fubini's theorem. So we can see that

$$u(x) = \frac{1}{(2\pi)^n} \int e^{i\lambda x} \hat{u}(\lambda) d\lambda.$$

which is a function because $\hat{u}$ is in $L^1(\mathbb{R}^n)$. It is also continuous, by using the same argument as in Lemma 4.6. □

Definition 4.16. *We say that $u \in D'(X)$ is in $H^s_{loc}(X)$ if ψu is in $H^s(\mathbb{R}^n)$ for all $\psi \in D(X)$. Note that $\psi u \in \epsilon'(X)$ as it has compact support and hence in $\mathcal{S}'(\mathbb{R}^n)$.*

5 Application of Fourier Transform

In this section we will consider the constant coefficients partial differential equations of the form $P(D)u = f$. We are interested in the regularity of the solution. If we know a distribution u solves $P(D)u = f$ and we know how smooth f is, can we tell how smooth u is?.

5.1 Elliptic Operators

Definition 5.1. *For a differential operator of order (or degree) N, we mean*

$$P(D) = \sum_{|\alpha| \leq N} C_\alpha D^\alpha.$$

Define the **principal symbol** *of P, written σ_P, by*

$$\sigma_P(\lambda) = \sum_{|\alpha|=N} C_\alpha \lambda^\alpha.$$

We say $P(D)$ is elliptic if $\sigma_P(\lambda) \neq 0$ for all $\lambda \in \mathbb{R}^n \backslash \{0\}$.

Lemma 5.2. *Let P be an elliptic operator of order N. Then there exists $R, C > 0$ such that*

$$|P(\lambda)| \geq C\langle\lambda\rangle^N, \ \forall |\lambda| > R.$$

Proof. On the sphere S^{n-1} we know $\sigma_P(\lambda) \neq 0$ and since S^{n-1} is compact, so we have a minimal, i.e. $\inf_{S^{n-1}} |\sigma_P(\lambda)| \geq C$. Then writing $\sigma_P(\lambda) = \sum_{|\alpha|=N} C_\alpha \lambda^\alpha$, we have

$$|\sigma_P(\lambda)| = |\lambda|^N \left| \sum_{|\alpha|=N} C_\alpha \left(\frac{\lambda}{|\lambda|}\right)^\alpha \right| = |\lambda|^N |\sigma_P(\lambda)|_{S^{n-1}} \geq C|\lambda|^N.$$

Now

$$\begin{aligned} |P(\lambda)| &\geq |\sigma_P(\lambda)| - |P(\lambda) - \sigma_P(\lambda)| \\ &\geq |\lambda|^N \left(C - \left| \frac{P(\lambda) - \sigma_P(\lambda)}{|\lambda|^N} \right| \right). \end{aligned}$$

But $P(\lambda) - \sigma_P(\lambda)$ has order $N-1$ and so there exists $R > 0$ large enough such that the second term in the bracket is bounded by, say $\frac{C}{2}$, and so the result follows. $\square$

Definition 5.3. *Given a constant coefficients differential operator $P(D)$, we say $E \in D'(\mathbb{R})$ is a parametrix for $P(D)$ if $P(D)E = \delta_0 + \omega$, where ω is a smooth functions.*

Lemma 5.4. *If $P(D)$ is elliptic with constant coefficients, then it has a parametrix that is smooth on $\mathbb{R}\backslash\{0\}$.*

Proof. By Lemma 5.2, we fix we constants $R, C > 0$ such that

$$|P(\lambda)| \geq C\langle\lambda\rangle^N.$$

(Note that $\langle\lambda\rangle$ is compatible with $|\lambda|$ for $|\lambda|$ large enough.) We shall introduce $\chi \in D(\mathbb{R}^n)$ such that $\chi(\lambda) = 1$ on $|\lambda| < R$. Define

$$\hat{E}(\lambda) = \frac{1 - \chi(\lambda)}{P(\lambda)}.$$

Then $\hat{E}$ is a smooth function (because $P(\lambda)$ is non-zero for $|\lambda| > R$) and has compact support, so $\hat{E} \in \mathcal{S}'(\mathbb{R}^n)$.

Using a similar result for distribution as in Lemma 4.7 (see Exercises 2), we have

$$(P(D)E)^\wedge(\lambda) = P(\lambda)\hat{E}(\lambda) = 1 - \chi(\lambda).$$

Taking Fourier inverse, we have

$$P(D)E = \delta_0 + \omega,$$

where $\hat{\omega} = -\chi \in \mathcal{S}(\mathbb{R}^n)$ and so $\omega \in \mathcal{S}(\mathbb{R}^n)$.

To show E is smooth away from 0, it is the same as showing $D^\alpha(x^\beta E)$ is continuous. We have

$$|(D^\alpha(x^\beta E))^\wedge| = |\lambda^\alpha D^\beta \hat{E}|.$$

It vanishes on $|\lambda| \leq R$, and for λ large, we can pick χ so that the contribution is bounded by a constant, and so it is of order

$$|\lambda^\alpha D^\beta P(\lambda)^{-1}| = O(|\lambda|^{|\alpha| - N - |\beta|}).$$

So we can pick $|\beta|$ large enough so that this is in $L^1(\mathbb{R}^n)$. Hence, for this β, $D^\alpha(x^\beta E)$ is continuous and so E is smooth away from 0. □

Lemma 5.5. *If $u \in \epsilon'(\mathbb{R}^n)$ and $P(D)u = f$ where $P(D)$ is elliptic of order N. If $f \in H^s(\mathbb{R}^n)$, then $u \in H^{N+s}(\mathbb{R}^n)$.*

Proof. Write $u = \delta_0 * u$. Then by previous lemma, we have

$$\delta_0 * u = [P(D)E - \omega] * u = P(D)E * u - \omega * u = E * P(D)u - \omega * u = E * f - \omega * u,$$

where we use Lemma 2.19 to deduce that $P(D)E * u = E * P(D)u$.

Now that $(E * f)^\wedge = \hat{E}\hat{f}$ (see Exercises 2), and the contribution of $\omega * u$ is small because it is in $\mathcal{S}(\mathbb{R}^n)$, so

$$\langle\lambda\rangle^{s+N}|\hat{u}(\lambda)| \lll \langle\lambda\rangle^{s+N}\hat{E}\hat{f} \lll \langle\lambda\rangle^{s}\hat{f},$$

because $\hat{E}(\lambda) \sim \frac{1}{P(\lambda)} \leq \frac{C}{\langle\lambda\rangle^N}$ for some constant C. (The advantage of using $\langle\lambda\rangle$ instead of $|\lambda|$ is that it can takes care of the case when $|\lambda|$ is small)

Thus,

$$\int \langle\lambda\rangle^{2(s+N)}|\hat{u}(\lambda)|^2 d\lambda < \|\hat{f}\|_{H^s(\mathbb{R}^n)} + C < \infty.$$

Hence $u \in H^{s+N}(\mathbb{R}^n)$. □

Theorem 5.6. *If $P(D)$ is elliptic constant coefficients differential operator of order N and $P(D)u = f$, where $f \in H^s_{loc}(\mathbb{R}^n)$. Then $u\ in H^{s+N}_{loc}(\mathbb{R}^n)$.*

Proof. Let $\psi \in D(X)$. Consider $\psi_0, \psi_1, \ldots, \psi_{M+1} \in D(X)$ with $\psi_{M+1} = \psi$ such that $\mathrm{supp}(\psi_i) \subset \mathrm{supp}(\psi_{i-1})$ and $\psi_{i=1} = 1$ on $\mathrm{supp}(\psi_i)$.

consider $\psi_0 u \in \epsilon'(\mathbb{R}^n)$, it is a fact (see Exercises 2) that there exists t such that $\psi_0 u \in H^t(\mathbb{R})$. Now we can write

$$P(D)(\psi_1 u) = \psi_1 P(D)u + [P(D), \psi_1](u) = \psi_1 f + [P(D), \psi_1](\psi_0 u),$$

because $\psi_0 = 1$ on $\mathrm{supp}(\psi_1)$. (Here $[P(D), \psi_1]$ is the commutator.) Since $\psi_1 f \in H^s(\mathbb{R}^n)$, $\psi_0 u \in H^t(\mathbb{R}^n)$ and the commutator $[P(D), \psi]$ has order $N-1$, so we conclude that

$$P(D)(\psi_1 u) \in H^{A_1}(\mathbb{R}^n), \text{ where } A_1 = \min\{s, t - N + 1\}.$$

But $\psi_1 u \in \epsilon'(\mathbb{R}^n)$ and hence by previous lemma,

$$\psi_1 u \in H^{B_1}(\mathbb{R}^n), \text{ where } B_1 = A_1 + N = \min\{s + N, t + 1\}.$$

Repeat these, so we have

$$B_M = \min\{s + N, t + M\}.$$

By picking M large enough we conclude that $\psi_M u \in H^{s+N}(\mathbb{R}^n)$. Finally, $\psi_M = 1$ on $\mathrm{supp}(\psi)$ and so the same argument applies and we have $\psi u \in H^{s+N}(\mathbb{R}^n)$. □

5.2 Fundamental Solutions

Definition 5.7. *If $E \in D'(\mathbb{R}^n)$ satisfies $P(D)E = \delta_0$, then we say E is a* **fundamental solution** *for $P(D)$.*

Example 5.8. *The Cauchy-Riemann operator*

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial x_2}\right).$$

has fundamental solution $E = \frac{1}{\pi z}$.

Note that E defines a distribution on $\mathbb{R}^2$ because it is locally integrable. We have

$$\begin{aligned}\left\langle \frac{\partial E}{\partial \bar{z}}, \psi \right\rangle &= -\int E\frac{\partial \psi}{\partial \bar{z}} dx \\ &= -\lim_{\epsilon \to 0} \int_{|x|>\epsilon} E\frac{\partial \psi}{\partial \bar{z}} dx \\ &= -\lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{\partial (E\psi)}{\partial \bar{z}} dx,\end{aligned}$$

because $\frac{\partial E}{\partial \bar{z}} = 0$ for all $|x| > \epsilon$. Now, by writing

$$dx = dx_1 \wedge dx_2 = \frac{1}{4i}(dz + d\bar{z}) \wedge (dz - d\bar{z}) = \frac{1}{2i} d\bar{z} \wedge dz,$$

or applying Green's theorem directly, the above is

$$\lim_{\epsilon \to 0} \frac{1}{2i} \oint_{|z|=\epsilon} E\psi dz,$$

because the contribution from infinity is 0. Let $z = \epsilon e^{i\theta}$, then we have, by DCT

$$\frac{1}{2i} \lim_{\epsilon \to 0} \int_0^{2\pi} \frac{1}{\pi \epsilon e^{i\theta}} \psi(\epsilon, \theta) i\epsilon e^{i\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \psi(0) d\theta = \langle \delta_0, \psi \rangle.$$

Hence E is a fundamental solution.

Example 5.9. *Consider the heat operator $\partial_t - \Delta_x$, with $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. Then*

$$E(x,t) = \begin{cases} (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} & t > 0, \\ 0 & t \le 0. \end{cases}$$

is a fundamental solution Clearly it is a distribution as it is locally integrable. By direct computation, we have

$$(\partial_t - \Delta_x)E = 0, \ \ on \ t > 0.$$

Now we have

$$\begin{aligned}
\langle(\partial_t - \Delta_x)E, \psi\rangle &= -\langle E, (\partial_t + \Delta_x)\psi\rangle \\
&= -\lim_{\epsilon\to 0}\int_\epsilon^\infty dt \int dx E(x,t)(\partial_t + \Delta_x)\psi \\
&= -\lim_{\epsilon\to 0}\int_\epsilon^\infty dt \int dx[\psi\Delta_x E - \psi\partial_t E] + \partial_t(\psi E) \\
&= -\lim_{\epsilon\to 0}\int_\epsilon^\infty dt \int dx \partial_t(\psi E) \\
&= \lim_{\epsilon\to 0}\int E(x,\epsilon)\psi(x,\epsilon)dx \\
&= \lim_{\epsilon\to 0}\int \frac{1}{(4\pi\epsilon)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\epsilon}}\psi(x,\epsilon)dx \\
&= \lim_{\epsilon\to 0}\int \pi^{-\frac{n}{2}} e^{|y|^2}\psi(2\sqrt{\epsilon}y,\epsilon)dy \\
&= \psi(0,0) = \langle\delta_0, \psi\rangle,
\end{aligned}$$

where we used the substitution $x = 2\sqrt{\epsilon}y$ *and DCT in the last line.*

Does every non-zero partial differential operator $P(D)$ has a fundamental solution? We will now motivate our approach. Try the construction

$$\langle E, \psi\rangle = \int \frac{\hat{\psi}(-\lambda)}{P(\lambda)} d\lambda.$$

Then we have

$$\begin{aligned}
\langle P(D)E, \psi\rangle &= \langle E, P(-D)\psi\rangle \\
&= \int \frac{(P(-D)\psi)^\wedge(-\lambda)}{P(\lambda)} d\lambda \\
&= \int \frac{P(\lambda)\hat{\psi}(-\lambda)}{P(\lambda)} d\lambda \\
&= \frac{1}{2\pi}\psi(0),
\end{aligned}$$

by inversion formula. So $P(D)E = \delta_0$. But this does not necessarily define an element in $D'(\mathbb{R}^n)$ because $\frac{1}{P(\lambda)}$ needs not be locally integrable.

Lemma 5.10. *For $x \in \mathbb{R}^n$, write $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Then for $\psi \in D(\mathbb{R}^n)$, we have $\hat{\psi}(\lambda', z)$ is a complex analytic function of $z \in \mathbb{C}$ for each $\lambda' \in \mathbb{R}^{n-1}$ and*

$$|\hat{\psi}(\lambda', z)| \leq C_m(1+|z|)^{-m} e^{\delta|\Im z|}, m = 0, 1, \ldots$$

for some constants δ and C_m where C_m depends on $\sup|\partial^\alpha \psi|$ for all $|\alpha| \leq m$.

Proof. Since $|z|^m$ is compatible with $(1+|z|)^m$, (again when $|z|$ is small we take care of the error by a constant and when $|z|$ is large, they are roughly the same) so consider

$$|z^m \hat{\psi}(\lambda', z)| \leq \int dx' \int dx_n e^{|x_n||\Im z|} \left| \frac{\partial^m}{\partial x_n^m} \psi(x', x_n) \right|,$$

where we used $z^m \hat{\psi}(\lambda', z) = (-iD^m_{x_n} \psi(x', x_n))^\wedge$. But $\psi \in D(\mathbb{R}^n)$, and so $|x_n|$ is bounded say by some δ. Also, the integral is finite and so is some constant. Hence the estimate follows either by using the above argument about $|z|^m$ and $(1+|z|)^m$ or use $(1+|z|)^m = \sum_{r=0}^m C_r|z|^r$.

For analyticity, recall that Morera's theorem tells that if a complex function has 0 integral around any closed contour, then the function is analytic. But

$$\oint_\gamma \hat{\psi}(\lambda', z)dz = \oint_\gamma \int dx' \int dx_n e^{-i\lambda' x' - ix_n z} \psi(x', x_n) dz,$$

which is 0 because ψ is a smooth function (and γ is closed). □

Corollary 5.11.

$$\int_{\Im z = C} \hat{\psi}(\lambda', z)dz = \int_{\Im z = 0} \hat{\psi}(\lambda', z)dz = \int_{\mathbb{R}} \hat{\psi}(\lambda', \lambda_n) d\lambda_n.$$

Proof. Use Cauchy theorem on the contour including $\Im z = C$ and $\mathbb{R}$ with the estimate in the previous lemma. □

Theorem 5.12. *Every non-constant coefficients differential operator admits a fundamental solution.*

Proof. By rotating and scaling coordinate, we can assume that $P(D)$ is given by

$$P(\lambda) = P(\lambda', \lambda_n) = \lambda_n^M + \sum_{i=0}^{M-1} a_i(\lambda')\lambda_n^i.$$

Fix $\mu' \in \mathbb{R}^{n-1}$, and consider $P(\mu', \lambda_n)$ as a polynomial in λ_n, write

$$P(\mu', \lambda_n) = \prod_{i=1}^M (\lambda_n - \tau_i(\mu')),$$

where $\tau_i(\mu')$ are roots of $P(\mu', \lambda_n)$.

We claim that in the complex plane, we can draw a horizontal line in the region $|\lambda_n| \le M + 1$ such that

$$|\Im\lambda_n - \Im\tau_i(\mu')| > \frac{1}{2} \ \forall i.$$

Indeed, we have $2M+2$ strips in this region and we have M roots, so at most M of them can be filled with the roots $\tau_i(\mu')$. Hence by pigeon hold principal, there must exist two adjacent stirps, both of which contain no roots $\tau_i(\mu')$. Then the line between the two strips is the one we want. On this line,

$$|P(\mu', \lambda_n)| > \prod_{i=1}^{M} \frac{1}{2} = 2^{-M}.$$

Since each $\tau_i(\mu')$ is a continuous function in μ', so for λ' in a small neighbourhood of μ', we also have the above inequality. For each μ', we set the line to be $\Im\lambda_n = c(\mu')$.

Now the argument works for any $\mu' \in \mathbb{R}^{n-1}$. We can cover all of $\mathbb{R}^{n-1}$ with small neighbourhoods (in fact countably many) such that for each $\lambda' \in \mathbb{R}^{n-1}$ there exists $c(\lambda')$ such that

$$|P(\lambda', \lambda_n)| \ge 2^{-M}$$

whenever $\Im\lambda_n = c(\lambda')$. This is because we can cover $\mathbb{R}^{n-1}$ by the balls $B(0, r)$ and each $B(0, r)$ is compact so when we use an infinite cover, $N(\lambda')$, where $N(\lambda')$ is a small neighbourhood of λ' such that the inequality holds. Then we have a finite cover for $B(0, r)$ and hence a countable cover for $\mathbb{R}^{n-1}$. So we can enumerate them as $N(\lambda'_1), N(\lambda'_2), \ldots$ and define

$$\Delta_i = N(\lambda'_i) \backslash \bigcup_{j=1}^{i-1} N(\lambda'_j) \ \forall i \ge 2.$$

So the sets Δ_i again cover $\mathbb{R}^{n-1}$ and they are disjoint.

Thus, by the previous discussion on our motivated method, we define

$$E = \int d\lambda' \int_{\mathbb{R}} \frac{\hat{\psi}(-\lambda', -\lambda_n)}{P(\lambda', \lambda_n)} d\lambda_n.$$

Then

$$E = \sum_{i=1}^{\infty} \int_{\Delta_i} d\lambda' \int_{\Im\lambda_n = c_i} \frac{\hat{\psi}(-\lambda', -\lambda_n)}{P(\lambda', \lambda_n)} d\lambda_n,$$

where we use Corollary 5.11 to translate each contour for $\lambda_n = \mathbb{R}$ to $\Im\lambda_n = c_i$ We check that $E \in D'(\mathbb{R}^n)$. Indeed, we have

$$\begin{aligned} |\langle E, \psi\rangle| &\leq \sum_{i=1}^{\infty} \int_{\Delta_i} d\lambda' \int_{\Im\lambda_n = c_i} \left| \frac{\hat{\psi}(-\lambda', -\lambda_n)}{P(\lambda', \lambda_n)} d\lambda_n \right| \\ &\leq 2^M \sum_{i=1}^{\infty} \int_{\Delta_i} d\lambda' \int_{\Im\lambda_n = c_i} |\hat{\psi}(-\lambda', -\lambda_n)| |d\lambda| \\ &\leq 2^M e^{\delta(M+1)} \sum_{i=1}^{k} \int_{\Delta_i} d\lambda' \int_{\Im\lambda_n = c_i} \langle\lambda_n\rangle^{-m} C_m d\lambda_n \\ &\leq C \sum_{|\alpha| \leq m} \sup |\partial^\alpha \psi|, \end{aligned}$$

where in the second last line we used Lemma 5.10 and the fact that each $|c_i| < M + 1$, and c_m depends on $\sup |\partial^\alpha \psi|$ for all $|\alpha| \leq m$. □

5.3 Structure Theorem

Theorem 5.13. *For each $u \in \epsilon'(X)$, there exists $f_\alpha \in C(X)$ with supp$(f_\alpha) \subset X$ and $u = \sum_\alpha \partial^\alpha f_\alpha$.*

Proof. Fix $\rho \in D(X)$ such that $\rho = 1$ on supp(u). Then for all $\psi \in \epsilon(X)$, we have

$$\langle u, \psi\rangle = \langle u, \rho\psi\rangle.$$

By extending u by 0, we can treat u as an element in $\epsilon'(\mathbb{R}^n)$. Now since $\rho\psi \in D(\mathbb{R}^n)$, there is $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\rho\psi = (\hat{\phi})^\wedge$. In other words, we have by Theorem 4.13, that

$$(2\pi)^n \check{\phi} = \rho\psi.$$

Also, we have

$$\langle u, \psi\rangle = \langle u, \rho\psi\rangle = \langle \hat{u}, \hat{\phi}\rangle.$$

For any $\phi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\hat{\phi} = \langle\lambda\rangle^{-2M} \left[(1 - \Delta)^M \phi\right]^\wedge,$$

because

$$[(1 - \Delta)\phi]^\wedge = (1 + |\lambda|^2)^M \hat{\phi} = \langle\lambda\rangle^{2M}.$$

Hence,

$$\langle u, \psi\rangle = \langle \hat{u}, \langle\lambda\rangle^{-2M} \left[(1 - \Delta)^M \phi\right]^\wedge\rangle = \langle \langle\lambda\rangle^{-2M} \hat{u}, \left[(1 - \Delta)^M \phi\right]^\wedge\rangle.$$

We have (see Exercises 2), for each $u \in \epsilon'(\mathbb{R}^n)$,

$$|\hat{u}(\lambda)| = |\langle u(x), e^{-i\lambda x}\rangle| \leq C \sum_{|\alpha|\leq N} \sup |\partial_x^\alpha e^{-i\lambda x}| \leq C' \langle\lambda\rangle^N.$$

So for M sufficiently large, we have $\langle\lambda\rangle^{-2M}\hat{u}(\lambda) \in L^1(\mathbb{R}^n)$. In fact the least such M has $-2M + ord(u) < -n$. Then for M large, define

$$f(x) = \frac{1}{(2\pi)^n}\int e^{i\lambda x}\langle\lambda\rangle^{-2M}\hat{u}(\lambda)d\lambda.$$

Then, from above we have

$$\begin{aligned}
\langle u, \psi\rangle &= \langle \hat{f}, \left[(1-\Delta)^M\phi\right]^\wedge\rangle \\
&= \langle (2\pi)^n\check{f}, (1-\Delta)^M\phi\rangle \\
&= \langle f, (1-\Delta)^M\left[(2\pi)^n\check{\phi}\right]\rangle \\
&= \langle f, (1-\Delta)^M(\rho\psi)\rangle.
\end{aligned}$$

Now we have

$$(1-\Delta)^M(\rho\psi) = \sum_\alpha (-1)^{|\alpha|}\rho_\alpha\partial^\alpha\psi,$$

where $\rho_\alpha \in D(X)$. Set $f\rho_\alpha = f_\alpha$, then

$$\langle u, \psi\rangle = \left\langle f, \sum_\alpha (-1)^{|\alpha|}\rho_\alpha\partial^\alpha\psi\right\rangle = \left\langle \sum_\alpha \partial^\alpha f_\alpha, \psi\right\rangle.$$

Hence, $u = \sum_\alpha \partial^\alpha f_\alpha$, where $f_\alpha \in C(X)$ by Lemma 4.6. □

Recall that if $u \in \epsilon'(\mathbb{R})$ then we have

$$\hat{u}(\lambda) = \langle u(x), e^{-i\lambda x}\rangle.$$

(see Exercises 2) This is real and analytic. We make a complex extension by allowing $\lambda \to z \in \mathbb{C}$. We call $\hat{u}(z)$ the Fourier Transform of $u \in \epsilon'(\mathbb{R})$ and define

$$\hat{u}(z) = \langle u(x), e^{-izx}\rangle, z \in \mathbb{C}\rangle.$$

Lemma 5.14. *If $u \in \epsilon'(\mathbb{R})$ and $supp(u) \subset [-\delta, \delta]$, then there exists $C, N > 0$ such that*

$$|\hat{u}(z)| \leq C(1+|z|)^N e^{\delta|\Im z|}\ \forall z \in \mathbb{C}.$$

Proof. Introduce $\psi \in C^\infty(\mathbb{R})$ such that $\psi = 1$ on $x \geq -\frac{1}{2}$ and $\psi = 0$ on $x \leq -1$. Now consider the function $\psi_\epsilon(x) = \psi(\epsilon(\delta - |x|))$, then we have

$$\psi_\epsilon(x) = \begin{cases} 1 & \text{if } |x| \leq \delta + \frac{1}{2\epsilon}, \\ 0 & \text{if } |x| \geq \delta + \frac{1}{\epsilon} \end{cases}$$

So $\psi_\epsilon(x) = 1$ on $\text{supp}(u)$.

$$\hat{u}(z) = \langle u(x), e^{-ixz} \rangle = \langle u(x), \psi_\epsilon(x) e^{-izx} \rangle.$$

Now since $u \in \epsilon'(\mathbb{R})$, we have $C, N > 0$ such that

$$|\hat{u}(z)| = |\langle u(x), \psi_\epsilon(x) e^{-ixz} \rangle| \leq C \sum_{|\alpha| \leq N} \sup |\partial_x^\alpha (\psi_\epsilon(x) e^{-izx})|.$$

But $|\partial^\beta \psi_\epsilon(x)| \leq C' \epsilon^\beta$ by chain rule, (as ψ_ϵ has compact support) and $|\partial_x^\gamma e^{-ixz}| \leq |z|^\gamma e^{|x||\Im z|}$. Hence we have

$$|\hat{u}(z)| \leq C'' \sum_{|\alpha| \leq N} \sum_{\beta + \gamma = \alpha} \epsilon^\beta |z|^\gamma e^{(\delta + \frac{1}{\epsilon})|\Im z|} \leq C'''(1 + |z|)^N e^{\delta |\Im z|},$$

by choosing $\epsilon = |z|$ and use the fact that $\psi_\epsilon(x) = 0$ for $|x| \geq \delta + \frac{1}{\epsilon}$. □

A similar result can be deduced for $\psi \in D(\mathbb{R}^n)$ with $\text{supp}(\psi) \subset [-\delta, \delta]$.

Lemma 5.15. *If $\psi \in D(\mathbb{R}^n)$ with supp$(\psi) \subset [-\delta, \delta]$, then*

$$|\hat{\psi}(z)| \leq C_m (1 + |z|)^{-m} e^{\delta |\Im z|}, m = 0, 1, 2, \ldots$$

Theorem 5.16. [Paley-Wiener-Schwarz]
(A) If $u(z)$ is an entire function of $z \in \mathbb{C}$ and

$$(a)\ |u(z)| \leq C_m (1 + |z|)^{-m} e^{\delta |\Im z|}, m = 0, 1, 2, \ldots$$

Then $u = \hat{\psi}$ for some $\psi \in D(\mathbb{R})$ with supp$(\psi) \subset [-\delta, \delta]$.
(B) If $U(z)$ is an entire function of $z \in \mathbb{C}$ and

$$(b)\ |U(z)| \leq C(1 + |z|)^N e^{\delta |\Im z|},$$

for some $C, N > 0$. Then $U = \hat{u}$ for some $u \in \epsilon'(\mathbb{R})$ with supp$(u) \subset [-\delta, \delta]$.

Proof. For $\lambda \in \mathbb{R}$, $u(\lambda) \in L^1(\mathbb{R})$, we define

$$\psi(x) = \frac{1}{2\pi} \int e^{i\lambda x} u(\lambda) d\lambda.$$

By differentiating the integral and using condition (a), we deduce that $\psi \in C^\infty(\mathbb{R}^n)$. Also, by using the estimate and Cauchy theorem, we have

$$\begin{aligned}
|\psi(x)| &= \left| \frac{1}{2\pi} \int_{\Im z=\eta} e^{ixz} u(z) dz \right| \\
&= \frac{1}{2\pi} \left| \int e^{i\lambda x} e^{-\eta x} u(\lambda + i\eta) d\lambda \right| \\
&\leq \frac{C_m}{2\pi} \int e^{-\eta x} e^{\delta|\eta|} (1+|\lambda|)^{-m} d\lambda \\
&\leq Ce^{\delta|\eta|-\eta x},
\end{aligned}$$

where we pick $m \geq 2$ so that the integral in the second last line converges and we use the obvious substitution $z = \lambda + i\eta$ and we have $|\lambda| < |z|$. Now we pick $\eta = tx$, where $t > 0$, and so we have

$$|\psi(x)| \leq Ce^{\delta|tx|-tx^2} = Ce^{t|x|(\delta-|x|)}.$$

This is true for all $t > 0$. If $|x| > \delta$, then we pick $t \to \infty$, which shows that $|\psi(x)| = 0$. Hence $\text{supp}(\psi(x)) \subset [-\delta, \delta]$.

For (B), firstly notice that $|U(\lambda)| \leq C(1+|\lambda|)^N$ for $\lambda \in \mathbb{R}$ and so $U \in \mathcal{S}'(\mathbb{R})$. So there exists $u \in \mathcal{S}'(\mathbb{R})$ such that $\hat{u} = U$. Introduce $\phi(x) \in D(\mathbb{R})$ such that $\text{supp}(\phi) \subset [-1, 1]$ and $\int \phi dx = 1$. Define

$$\phi_\epsilon(x) = \frac{1}{\epsilon} \phi\left(\frac{x}{\epsilon}\right),$$

and so $\phi_\epsilon \to \delta_0$ as $\epsilon \to 0$. Consider $u_\epsilon = u * \psi_\epsilon$. So $\hat{u}_\epsilon = \hat{u}\hat{\psi}_\epsilon = U\hat{\psi}_\epsilon$. Note that $\text{supp}(\psi_\epsilon) \subset [-\epsilon, \epsilon]$. So by Lemma 5.15, we have

$$|\hat{\psi}_\epsilon(z)| \leq C_m(\epsilon)(1+|z|)^{-m} e^{\epsilon|\Im z|}, m = 0, 1, 2, \ldots$$

So the analytic function $\hat{u}_\epsilon(z)$ obeys

$$|\hat{u}_\epsilon(z)| \leq C'(1+|z|)^{-m+N} e^{(\delta+\epsilon)|\Im z|}, m = 0, 1, 2, \ldots$$

Thus, by (A) this means $\text{supp}(u_\epsilon) \subset [-\delta-\epsilon, \delta+\epsilon]$. Finally, as $u_\epsilon \to u$ we have $\text{supp}(u) \subset [-\delta, \delta]$. □

6 Oscillatory Integral

In this section we shall study the integral of the forms

$$\int e^{i\Phi(x,\theta)} a(x,\theta)d\theta, (x,\theta) \in X \times \mathbb{R}^k,$$

where Φ is a phase function and a belongs to a class of functions called symbols. Our aim in this section is to develop tools so that we can understand and manipulate these objects in a meaningful way.

The definition will not be in terms of Riemann or Lesbegue integral because the integral may not exist in these settings. We will allow the function $a(\cdot,\theta)$ to grow as $|\theta| \to \infty$.

Before we introduce the formal definition of a phase function and of the class of symbols, to which a shall belong, we will develop some intuition for oscillatory integrals.

Example 6.1. *For $\chi \in D(\mathbb{R})$, we know*

$$e^{ix\theta}\chi(\theta)d\theta, \text{ where } \Phi(x,\theta) = x\theta$$

has rapid decays as $|x| \to \infty$. We proved this by letting $L = \frac{1}{ix}\frac{d}{d\theta}$, so that

$$Le^{i\theta x} = e^{i\theta x}.$$

Then

$$\int e^{ix\theta}\chi(\theta)d\theta = \int (L^N e^{ix\theta})\chi(\theta)d\theta = \int e^{ix\theta} L^{*N}\chi(\theta)d\theta = O\left(\frac{1}{|x|^N}\right),$$

where L^* is the formal adjoint of L.

Example 6.2. *Generalised Fourier Integrals. Set $\Phi(x,\theta) = xp(\theta)$, and let $L = \frac{1}{ixp'(\theta)}\frac{d}{d\theta}$. Then we have*

$$\int e^{ixp(\theta)}\chi(\theta)d\theta = \int L^N e^{ixp(\theta)}\chi(\theta)d\theta = \int e^{ixp(\theta)} L^{*N}\chi(\theta)d\theta.$$

This breaks down if $p'(\theta) = 0$ somewhere on supp(χ). This suggests the largest contribution of the above integral comes from a neighbourhood of the points for which $p'(\theta) = 0$. These points are called the points of stationary phase.

Lemma 6.3. *Let* $\Phi \in C^\infty(\mathbb{R})$ *with* $\Phi'(\theta) \neq 0$ *on* $\mathbb{R}\backslash\{0\}$ *and* $\Phi(0) = \Phi'(0) = 0$ *and* $\Phi''(0) \neq 0$. *Then for any* $\chi \in D(\mathbb{R})$ *we have*

$$\int e^{ix\Phi(\theta)}\chi(\theta)d\theta = O\left(\frac{1}{|x|^{\frac{1}{2}}}\right), |x| \to \infty.$$

Proof. Fix $\rho \in D(\mathbb{R})$ with $\mathrm{supp}(\rho) \subset [-1,1]$ and $\rho(\theta) = 1$ when θ is small enough. Also fix $\delta \in (0,1)$. Write

$$\begin{aligned}
I(x) &= \int \chi(\theta)\left[\rho\left(\frac{\theta}{\delta}\right) + 1 - \rho\left(\frac{\theta}{\delta}\right)\right] e^{ix\Phi(\theta)}d\theta \\
&= \int \rho\left(\frac{\theta}{\delta}\right)\chi(\theta)e^{ix\Phi(\theta)}d\theta + \int\left(1 - \rho\left(\frac{\theta}{\delta}\right)\right)\chi(\theta)e^{ix\Phi(\theta)}d\theta \\
&= I_1(x) + I_2(x).
\end{aligned}$$

Note that $\mathrm{supp}\big(\rho\left(\frac{\theta}{\delta}\right)\big) \subset (-\delta, \delta)$, and so

$$|I_1(x)| \leq C\int_{-\delta}^{\delta}|\chi| \leq C\delta.$$

For $I_2(x)$, since $\Phi'(\theta) \neq 0$ on the support of $\left(1 - \rho\left(\frac{\theta}{\delta}\right)\right)\chi(\theta)$, because 0 is not in the support of the function. Introduce the operator

$$L = \frac{1}{ix\Phi'(\theta)}\frac{d}{d\theta}.$$

Then on integrating by part, we have

$$I_2(x) = \int L^N e^{ix\Phi(\theta)}\left(1 - \rho\left(\frac{\theta}{\delta}\right)\right)\chi(\theta)d\theta.$$

By hypothesis, $\Phi'(\theta) = \theta\eta(\theta)$ where $\eta(0) \neq 0$ on $\mathbb{R}$. So on $\mathrm{supp}(\chi)$, there exists c such that

$$|\Phi'(\theta)| \geq c|\theta|.$$

Now by chain rule, since the formal adjoint

$$L^* = (-1)\frac{d}{d\theta}\left(\frac{1}{ix\Phi'(\theta)}, \cdot\right).$$

So we have

$$\left|L^{*N}\left(1 - \rho\left(\frac{\theta}{\delta}\right)\right)\chi(\theta)\right| \leq Cx^{-N}|\theta|^{-N}|1 - \delta^{-N}|.$$

This is because $\frac{1}{\Phi'(\theta)}| \leq \frac{c}{|\theta|}$, and the fact that the maximum we have is the worst we get, differentiate $\left(1 - \rho\left(\frac{\theta}{\delta}\right)\right)\chi(\theta)$ N times and we also have

$$\frac{d^{(n)}}{d\theta}\left(\rho\left(\frac{\theta}{\delta}\right)\right) = \frac{1}{\delta^n}\rho^{(n)}\left(\frac{\theta}{\delta}\right).$$

Also we know that $|\chi^{(n)}(\theta)|$ is bounded and $\rho^{(N)}$ is bounded as they belong to $D(\mathbb{R})$.

So when we integrate over $J_\delta = \{\theta : |\theta| > \delta\} \cap \operatorname{supp}(x)$, we have

$$I_2(x) \leq C \int_{|\theta|>\delta} |1 - \delta^{-N}||\theta|^{-N}|x|^{-N} d\theta \leq C|x|^{-N}\delta^{-2N+1}.$$

Also, for the part $|\theta| < \delta$, it is again easy to estimate. Hence, combining I_1 and I_2, we have

$$\left|\int e^{ix\Phi(\theta)}\chi(\theta)d\theta\right| \leq C \max\{\delta, |x|^{-N}\delta^{-2N+1}\}.$$

The right hand side is smallest when $\delta = |x|^{-\frac{1}{2}}$ and so the result follows. □

Definition 6.4. *Let $X \subset \mathbb{R}^n$ be open. A smooth function $a : X \times \mathbb{R}^k \to \mathbb{C}$ is called a symbol of order N if for each compact $K \subset X$ and for each pair of multi-indices α, β there exists a constant C such that*

$$|D_x^\alpha D_\theta^\beta a(x, \theta)| \leq C\langle\theta\rangle^{N-|\beta|},$$

for all $(x, \theta) \in X \times \mathbb{R}^k$. We denote the set of all symbols $sym(x, \mathbb{R}^k; N)$.

Example 6.5. *If $a(x, \theta) = p(\theta)$, p is a polynomial in θ of order N, then it defines an element of $sym(X, \mathbb{R}^k; N)$. We only really care about the large θ behavior of our symbols because we can always write $a = a_1 + a_2$ where θ-support of a_1 is compact.*

Definition 6.6. *A phase function is a function $\Phi : X \times \mathbb{R}^k \to \mathbb{R}$ such that*

(i) Φ is continuous on $X \times \mathbb{R}^k$ and homogeneous of order 1 in θ, i.e. $\Phi(x, \tau\theta) = \tau\Phi(x, \theta)$.

(ii) Φ is smooth on $X \times (\mathbb{R}^k \backslash \{0\})$.

(iii) $d\Phi = \nabla_x \Phi dx + \nabla_\theta \Phi d\theta$ is non-vanishing on $X \times (\mathbb{R}^k \backslash \{0\})$.

Example 6.7. *The function*

$$\frac{1}{(2\pi)^n}\int \theta^\alpha e^{ix\theta}d\theta$$

is an oscillatory integral with phase function $x\theta$. We check that, clearly, $x\theta$ is continuous and homogenous of order 1 in θ. It is smooth everywhere. Also, we can compute the derivative directly that

$$d\Phi = \left(\frac{\partial\Phi}{\partial x_1},\ldots,\frac{\partial\Phi}{\partial x_n},\frac{\partial\Phi}{\partial\theta_1},\ldots,\frac{\partial\Phi}{\partial\theta_n}\right) = (\theta_1,\ldots,\theta_n,x_1,\ldots,x_n)$$

which is non-vanishing at $\theta \neq 0$.

Lemma 6.8. *If $a \in sym(X,\mathbb{R}^k;N)$ then $D_x^\alpha D_\theta^\beta a \in sum(X,\mathbb{R}^k,N-|\beta|)$. If a_1,a_2 are symbols in $sym(X,\mathbb{R}^k;N)$ and $sym(X,\mathbb{R}^k;N_2)$ respectively, then $a_1a_2 \in sym(X,\mathbb{R}^k;N_1+N_2)$.*

Proof. See Exercises 3. □

We would like to define our oscillatory integral

$$I_\Phi(a) = \int e^{i\Phi(x,\theta)}a(x,\theta)d\theta$$

as a linear function on $D(X)$ via

$$\psi \to \langle I_\Phi(a),\psi\rangle = \int\int e^{i\Phi(x,\theta)}a(x,\theta)\psi(x)d\theta dx$$

Unfortunately this is not good enough because the double integral does not converge absolutely So instead we define $I_\Phi(a)$ to be the limit of

$$I_{\Phi,\epsilon}(a) = \int e^{i\Phi(x,\theta)}a(x,\theta)\chi(\epsilon\theta)d\theta,$$

where $\chi \in \mathbb{R}^k$ is fixed and $\chi = 1$ on $|\theta| < 1$ and $\chi = 0$ on $|\theta| > 2$.

Lemma 6.9. *There exists a differential operator*

$$L = \sum_{j=1}^k a_j(x,\theta)\frac{\partial}{\partial\theta_j} + \sum_{j=1}^n b_j(x,\theta)\frac{\partial}{\partial x_j} + c(x,\theta)$$

with $a_j \in sym(X,\mathbb{R}^k;0)$ and $b_j,c \in sym(X,\mathbb{R}^k;-1)$ and the formal adjoint L^* satisfies $L^*e^{i\Phi} = e^{i\Phi}$.

Proof. Consider the operator

$$T = -i \sum_{i=1}^{k} |\theta|^2 \frac{\partial \Phi}{\partial \theta_j} \frac{\partial}{\partial \theta_j} - i \sum_{j=1}^{n} \frac{\partial \Phi}{\partial x_j} \frac{\partial}{\partial x_j}.$$

Then we have

$$Te^{i\Phi(x,\theta)} = \frac{1}{\pi(x,\theta)} e^{i\Phi(x,\theta)},$$

where $\pi(x,\theta) = (|\theta|^2 |\nabla_\theta \Phi|^2 + |\nabla_x \Phi|^2)^{-1}$. Clearly $\pi \in C^\infty(X \times (\mathbb{R}^k \backslash \{0\}))$.

Since Φ is homogeneous of order 1 and note that

$$\tau \frac{\partial}{\partial x_j} \Phi(x,\theta) = \frac{\partial}{\partial x_j} \Phi(x, \tau\theta).$$

So $\frac{\partial \Phi}{\partial x_j}$ is homogenous of order 1. Similarly we can check that $\frac{\partial \Phi}{\partial \theta_j}$ is homogeneous of order 0 because

$$\frac{\partial \Phi}{\partial \tau\theta}(x, \tau\theta) = \frac{\partial \Phi}{\partial \theta}(x,\theta).$$

It follows that $\frac{1}{\pi(x,\theta)}$ is homogeneous of order -2 in θ.

Now define $L'^* = \pi(x,\theta)T$. Then we have

$$L'^* e^{i\Phi} = e^{i\Phi}.$$

Set $L^* = (1 - \rho(\theta))L'^* + \rho(\theta)$, where $\rho \in D(\mathbb{R}^n)$ has

$$\rho(\theta) = \begin{cases} 1 & \text{if } |\theta| < 1, \\ 0 & \text{if } |\theta| > 2. \end{cases}$$

Then clearly

$$L^* e^{i\Phi} = e^{i\Phi}.$$

Let L be the formal adjoint of L^*. In fact by defining L^* in such a way, we can see that L^* is smooth in the sense that each term a_j, b_j, c are smooth. To check L has the designed property, see Exercises 3. □

Following previous discussion, we shall prove the following theorem:

Theorem 6.10. *If Φ is a phase function and $a \in sym(X, \mathbb{R}^k; N)$. Then the limit*

$$I_\Phi(a) = \lim_{\epsilon \to 0} I_{\Phi,\epsilon}(a)$$

defines an element of $D'(X)$ of order at most $N + K + 1$.

Proof. For each $\epsilon > 0$, we have

$$\begin{aligned}
\langle I_{\Phi,\epsilon(a)}, \psi\rangle &= \int\int e^{i\Phi(x,\theta)} a(x,\theta)\chi(\epsilon\theta)\psi(x) d\theta dx \\
&= \int\int (L^*)^M e^{i\Phi(x,\theta)} a(x,\theta)\chi(\epsilon\theta)\psi(x) d\theta dx \\
&= \int e^{i\Phi(x,\theta)} L^M[a(x,\theta)\chi(\epsilon\theta)\psi(x)] d\theta dx,
\end{aligned}$$

where L is as in the previous lemma. Note that if $a \in \mathrm{sym}(X, \mathbb{R}^k; N)$ then,

$$L^M[a(x,\theta)\psi(x)] = \sum_{|\alpha|\le M} a_\alpha(x,\theta)\partial^\alpha\psi, a_\alpha(x,\theta) \in \mathrm{sym}(X, \mathbb{R}^k; N-M).$$

(See Exercises 3 for the proof) Also, observe that

$$\begin{aligned}
|\partial^\alpha \chi(\epsilon\theta)| &= \epsilon^{|\alpha|}|\partial^\alpha_{\epsilon\theta}\chi(\epsilon\theta)| \\
&\le \epsilon^{|\alpha|} C_\alpha \langle\epsilon\theta\rangle^{-|\alpha|} \\
&= C_\alpha(\epsilon^{-2} + |\theta|^2)^{-\frac{|\alpha|}{2}} \\
&\le C_\alpha\langle\theta\rangle^{-|\alpha|},
\end{aligned}$$

for $0 \le \epsilon \le 1$, where in the second line we use the fact that $\chi(\theta) = 1+\theta^k\eta_k(\theta)$, and η_k is smooth, for all $k \ge 1$. Thus, $\chi(\epsilon\theta)$ is a symbol of order 0. So We can treat $a(x,\theta)\chi(\epsilon\theta)$ as a symbol of order N (see Exercises 3), uniformly in ϵ. Therefore, we have

$$L^M[a(x,\theta)\chi(\epsilon\theta)\psi(x)] = \sum_{|\alpha|\le M} a_\alpha(x,\theta;\epsilon)\partial^\alpha\psi, a_\alpha(x,\theta;\epsilon) \in \mathrm{sym}(X, \mathbb{R}^k; N-M).$$

So we can choose M large enough so that the θ integral is absolutely convergent, using $a \in \mathrm{sym}(X, \mathbb{R}^k; N-M)$ implies

$$|a(x,\theta)| \le C\langle\theta\rangle^{N-M}.$$

For example, we can take $M \ge N+k+1$, i.e. $N-M < -k$. Then we can apply DCT on the limit and hence

$$\begin{aligned}
\langle I_\Phi(a), \psi\rangle &= \int\int e^{i\Phi(x,\theta)} L^M[a(x,\theta)\psi(x)] d\theta dx \\
&= \int\int (L^M)^* e^{i\Phi(x,\theta)} a(x,\theta)\psi(x) d\theta dx \\
&= \int\int a(x,\theta)\psi(x) e^{i\Phi(x,\theta)} d\theta dx
\end{aligned}$$

Then, Recall that Theorem 2.9 says if each $u_k \in D'(X)$ and the limit $u = \lim_{k\to\infty} u_k$ exists then $u \in D'(X)$. So $I_\Phi(a) \in D'(X)$.

Finally, we have

$$\begin{aligned} |\langle I_\Phi(a), \psi\rangle| &\leq \int\int |L^M[a(x,\theta)\psi(x)]d\theta dx \\ &\leq \sum_{|\alpha|\leq M} |\partial^\alpha \psi| \int\int |a_\alpha(x,\theta)|d\theta dx \\ &\leq C \sum_{|\alpha|\leq N+k+1} \sup |\partial^\alpha \psi|, \end{aligned}$$

where we used $M = N + k + 1$ and the x integral converges because ψ has compact support. Hence $I_\Phi(a)$ has order at most $N + k + 1$. □

Now that we have shown $I_\Phi(a) \in D'(X)$, it is normal to ask when it is smooth.

Definition 6.11. *The* **singular support** *of a distribution, denoted singsup-p, is defined as the complement of the union of all the open sets on which u is smooth. For example, singsupp(δ) = $\{0\}$.*

Theorem 6.12. *For an oscillatory integral $I_\Phi(a)$ we have*

$$singsupp(I_\Phi(a)) \subset \{x : \nabla_x \Phi(x,\theta) = 0 \text{ for some } \theta \in \big(\mathbb{R}^k \cap supp(a(x,\theta))\big)\}.$$

Proof. Take $x_0 \in X$ such that $\nabla_\theta \Phi(x,\theta) \neq 0$ for any θ. Then by continuity, there is a small neighbourhood of x_0 in X such that $\nabla_\theta \Phi(x,\theta) \neq 0$.

Let $\psi \in D(X)$ be supported on this region and consider

$$\psi I_\Phi(a) = \int e^{i\Phi(x,\theta)} a(x,\theta)\psi(x) d\theta,$$

and we will write $a'(x,\theta)$ for $a(x,\theta)\psi(x)$. It is clear that $a' \in \mathrm{sym}(X, \mathbb{R}^k; N)$. Define the operator

$$L^* = \frac{1}{|\nabla_\theta \Phi|^2} \sum_{j=1}^{k} \frac{\partial \Phi}{\partial \theta_j} \frac{\partial}{\partial \theta_j}.$$

Then $L^* e^{i\Phi} = e^{i\Phi}$ and L^*, L are both well defined on supp(a').

Now L has the property that $L(a') \in \mathrm{sym}(X, \mathbb{R}^k, N-1)$. So

$$\begin{aligned} \langle \psi I_\Phi(a), \phi\rangle &= \lim_{\epsilon\to 0} \int\int e^{i\Phi(x,\theta)} a'(x,\theta)\chi(\epsilon\theta)\phi(x) d\theta dx \\ &= \lim_{\epsilon\to 0} \int\int e^{i\Phi(x,\theta)} L^M[a'(x,\theta)\chi(\epsilon\theta)]\phi(x) d\theta dx. \end{aligned}$$

Again $L^M[a'(x,\theta)\chi(\epsilon\theta)] \in \mathrm{sym}(X, \mathbb{R}^k; N-M)$ uniformly in $\epsilon < 1$ (because it's the sum of those). Hence for M large enough use DCT and so the θ integral converges, and since a' has support on which $\nabla_\theta \Phi(x,\theta) \neq 0$ and so the θ integral is smooth. Hence we conclude that $\psi I_\Phi(a)$ is smooth everywhere, and so $I_\Phi(a)$ is smooth at x_0 for any $\nabla_\theta \Phi(x,\theta) \neq 0$ (because ψ is supported there). □

Example 6.13. *Consider $X = \mathbb{R}^n$ and $k = n$, the oscillatory integral*

$$D^\alpha \delta_0(x) = \frac{1}{(2\pi)^n} \int e^{ix\theta} \theta^\alpha d\theta.$$

This has phase function $\Phi(x,\theta) = x\theta$. So by the previous theorem, we indeed have

$$singsupp D^\alpha \delta_0(x) \subset \{0\}.$$

Example 6.14. *Consider the Cauchy problem*

$$\frac{\partial u}{\partial t} + \underline{C}\nabla_x u = 0, u(\underline{x}, 0) = \delta_0(\underline{x}), \underline{C} \in \mathbb{R}^n.$$

Using the coordinate $x = (\underline{x}, t)$, and apply Fourier Transform in the $\underline{x}$ coordinate, we have

$$\frac{\partial \hat{u}(\underline{\lambda}, t)}{\partial t} + i\underline{C} \cdot \underline{\lambda}\hat{u} = 0, \hat{u}(\underline{\lambda}, 0) = 1.$$

It is easy to check that

$$\hat{u}(\underline{\lambda}, t) = \hat{u}(\underline{\lambda}, 0)e^{-i\underline{C}\cdot\underline{\lambda}t}.$$

Then we have

$$u(\underline{x}, t) = \frac{1}{(2\pi)^n} e^{i\underline{\lambda}\cdot\underline{x} - i\underline{C}\cdot\underline{\lambda}t} d\underline{\lambda}.$$

Then the previous theorem tells that

$$singsupp(u) \subset \{(\underline{x}, t) : \nabla_\lambda[\underline{\lambda}(\underline{x} - \underline{C}t)] = 0\} = \{(\underline{x}, t) : \underline{x} = \underline{C}t\}.$$

The solution of general problem with $u(x, 0) = f(x) \in \epsilon'(\mathbb{R})$ can be obtained with convolution, i.e. $U = u * f$.